A Brief Introduction to Inequalities

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Prologue

One morning when I was in 4th grade, I remember waking up at 5AM to travel with my family to the far side of Puerto Rico. We were on a 3 hour drive to Mayagüez to participate in the second round of a math test given at the University of Puerto Rico (UPR), Mayagüez campus. I remember arriving and seeing hundreds of students that were going to take this test. It was all very exciting to see.

During that visit, my parents met Dr. Luis Cáceres and Dr. Arturo Portnoy, professors at the university and in charge of the contest. My parents have said that this simple meeting helped launch my math career because with only a few words of encouragement they were able to learn some basic information to gather resources so I could feed my interest for math.

The first interesting inequality questions I remember seeing were given to me by Cornel Pasnicu. It was during the MathCounts State round competition in 7th grade and he was challenging me with different problems. As I began to work on them I noticed that many inequality problems can be stated simply but are very difficult to answer. The first two example problems in this book are those two that Cornel had given me. Having worked many hours over the past 6 years preparing for various math olympiads, inequality questions are the most fun for me.

In 11th grade, a friend asked me to write a short lecture on Inequalities for a website he was creating. After finishing the lecture I posted a link to it online where Dr. Arturo Portnoy read it and recommended I give the lecture at the upcoming OMPR Saturday class, and so I did. This was a huge honor for me but I was quite nervous, having to stand up in front of friends knowing that high school students had never given these classes before. I asked my friend Gabriel Reilly to help me and judging from the feedback we received, it was a great success. That lecture became the basis for this book which I hope students preparing for math olympiads can use.

And finally, there have been many people in my life that have helped to advance my love for math. I have already mentioned Dr. Cáceres, Dr. Portnoy and Dr. Pasnicu, who have helped and inspired me more than I can put into words. But Dr. Portnoy deserves a special mention here as he has helped with the proofing of this book.

Another math professor that has inspired me is Dr. Francis Castro at the UPR Río Piedras campus. When I was in 8th grade he invited me to take university level pre-calculus at UPR during the summer. Dr. Castro has for many years gone out of his way to present me with challenging math problems and I will always be grateful for his interest in my career.

The best math coach ever award goes to professor Nelson Ciprián from Colegio Espíritu Santo (CES). For many years CES and Mr. Ciprián have produced the top high school math talent in all of Puerto Rico. He has been my math coach for 6 years and I will always be thankful for his guidance.

Over the years brother Roberto Erb, aunts like Rosemary Erb, uncles, grandparents and family friends like Dr. Yolanda Mayo, The Reilly's and many others have helped sponsor the math camps I have attended. Without their help I wouldn't have been able to get to math camps like Awesome Math. And finally, I want to thank my family.

My mom for always being there to support me.

My dad for always inspiring me to do greater.

Contents

Chapter 1

The Basics

1.1 A Trivial Inequality

Take any real number, say x for example, and square it. No matter what x you choose, the result, x^2, is always non-negative (i.e. $x^2 \geq 0$). This is known as the Trivial Inequality and is the base for many inequality problems.

When attempting to use this inequality, try to rearrange the problem so that there is a zero on the right hand side and then factor the expression on the left hand side in a way that it's made up of "squares".

Example 1.1.1: Let a and b be real numbers. Prove that

$$a^2 + b^2 \geq 2ab$$

Proof. Note that by subtracting $2ab$ on both sides we get

$$a^2 - 2ab + b^2 \geq 0$$

or

$$(a - b)^2 \geq 0$$

which is true due to the Trivial Inequality. Since both inequalities are equivalent, we are done. □

Example 1.1.2: Let a, b and c be real numbers. Prove that

$$a^2 + b^2 + c^2 \geq ab + bc + ac$$

Proof. We start by moving all of the terms to the left

$$a^2 + b^2 + c^2 - ab - bc - ac \geq 0.$$

By multiplying by 2 we can see that

$$2(a^2 + b^2 + c^2 - ab - bc - ac) = (a-b)^2 + (a-c)^2 + (b-c)^2 \geq 0.$$

Thus our original inequality is true, since both inequalities are equivalent. Alternatively, you could notice, from Example 1.1, that the following inequalities are true

$$\begin{aligned} a^2 + b^2 &\geq 2ab \\ b^2 + c^2 &\geq 2bc \\ a^2 + c^2 &\geq 2ac \end{aligned}$$

Hence their sum,

$$2(a^2 + b^2 + c^2) \geq 2(ab + bc + ac)$$

is also true, so all that is left is to do is divide by 2 and we're done. □

1.1.1 Useful Identities

When working with inequalities, it's very important to keep these identities in mind:

- $a^2 - b^2 = (a+b)(a-b)$
- $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
- $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc)$
- $abc = (a+b+c)(ab+bc+ac) - (a+b)(b+c)(a+c)$

1.1.2 Practice Problems

1. Let x be a non-zero real number. Prove that

   I.
   $$x^2 + 1 \geq 2x$$

   II.
   $$4x^2 + 1 \geq 4x$$

   III.
   $$x^2 + \frac{1}{x^2} \geq 2$$

2. Let a and b be real numbers. Prove that

   I.
   $$a^2 + 4b^2 \geq 4ab$$

   II.
   $$a^2 + b^2 + 1 \geq ab + a + b$$

   III.
   $$(a+b)^2 + 2a^2 + (a-b)^2 \geq 2b^2$$

   IV.
   $$a^2 - ab + b^2 \geq 0$$

   V.
   $$2(a^2 + b^2) \geq (a+b)^2 \geq 4ab$$

3. Let a and b be positive real numbers. Prove that

   I.
   $$(a+b)(1+ab) \geq 4ab$$

   II.
   $$a + b + 1 \geq 2\sqrt{a+b}$$

   III.
   $$(a+1)(b+1)(1+ab) \geq 8ab$$

   IV.
   $$(a^2 - b^2)(a-b) \geq 0$$

V.

$$\frac{(a^3 - b^3)(a - b)}{3} \geq ab(a - b)^2$$

4. (Grade 8 Romanian National Math Olympiad, 2008) (Part a) Prove that for all positive reals u, v, x, y the following inequality takes place:

$$\frac{u}{x} + \frac{v}{y} \geq \frac{4(uy + vx)}{(x + y)^2}$$

1.1.3 Solutions

1. Let x be a non-zero real number. Prove that

I.

$$x^2 + 1 \geq 2x$$

Proof. Subtract $2x$ on both sides and you are left with

$$x^2 - 2x + 1 \geq 0$$

which is equivalent to

$$(x-1)^2 \geq 0$$

a direct result of the Trivial Inequality. □

II.

$$4x^2 + 1 \geq 4x$$

Proof. Subtract $4x$ on both sides and you're left with

$$4x^2 - 4x + 1 \geq 0$$

which factorizes into

$$(2x-1)^2 \geq 0$$

and we're done! □

III.

$$x^2 + \frac{1}{x^2} \geq 2$$

Proof. Rewrite the inequality as

$$x^2 - 2 + \frac{1}{x^2} \geq 0$$

then, note that it is equivalent to

$$\left(x - \frac{1}{x}\right)^2 \geq 0$$

which is true. □

2. Let a and b be real numbers. Prove that

I.
$$a^2 + 4b^2 \geq 4ab$$

Proof. As before, we subtract the terms on the right hand side ($4ab$ in this case)
$$a^2 - 4ab + 4b^2 \geq 0$$
which is equivalent to
$$(a - 2b)^2 \geq 0$$
□

II.
$$a^2 + b^2 + 1 \geq ab + a + b$$

Proof. Note that in one of the examples we proved that
$$a^2 + b^2 + c^2 \geq ab + bc + ac$$
is true for all real numbers a, b and c. In this case we have that $c = 1$, hence this inequality is also true. In the same way, we conclude that
$$a^2 + b^2 + 1 \geq ab + a + b \Longleftrightarrow (a - b)^2 + (a - 1)^2 + (b - 1)^2 \geq 0$$
□

III.
$$(a + b)^2 + 2a^2 + (a - b)^2 \geq 2b^2$$

Proof. Rewrite the inequality as
$$(a + b)^2 + 2(a^2 - b^2) + (a - b)^2 \geq 0$$
Then note that if $x = a+b$ and $y = a-b$ then our inequality is equivalent with
$$x^2 + 2xy + y^2 \geq 0$$
or
$$(x + y)^2 \geq 0$$
and so we are done! □

IV.

$$a^2 - ab + b^2 \geq 0$$

Proof. Multiply both sides by 2 and rewrite as

$$a^2 + b^2 + (a^2 - 2ab + b^2) \geq 0$$

which is equivalent to

$$a^2 + b^2 + (a - b)^2 \geq 0.$$

The last inequality is of course a sum of squares, so we are done. □

V.

$$2(a^2 + b^2) \geq (a + b)^2 \geq 4ab$$

Proof. We'll first prove the left hand side inequality

$$\begin{aligned} 2(a^2 + b^2) &\geq (a + b)^2 \\ 2(a^2 + b^2) - (a + b)^2 &\geq 0 \\ 2(a^2 + b^2) - (a^2 + 2ab + b^2) &\geq 0 \\ a^2 - 2ab + b^2 &\geq 0 \\ (a - b)^2 &\geq 0 \end{aligned}$$

and so we have proven the left hand side of the inequality. For the right hand side we have

$$\begin{aligned} (a + b)^2 &\geq 4ab \\ (a + b)^2 - 4ab &\geq 0 \\ (a^2 + 2ab + b^2) - 4ab &\geq 0 \\ a^2 - 2ab + b^2 &\geq 0 \\ (a - b)^2 &\geq 0 \end{aligned}$$

thus both sides are solved. □

3. Let a and b be positive real numbers. Prove that

I.

$$(a+b)(1+ab) \geq 4ab$$

Proof. We'll start by proving the following two simpler inequalities

$$a+b \geq 2\sqrt{ab} \tag{1.1}$$
$$1+ab \geq 2\sqrt{ab} \tag{1.2}$$

The first holds since it is equivalent with

$$(\sqrt{a}-\sqrt{b})^2 \geq 0$$

while the second also holds since it is equivalent with

$$(1-\sqrt{ab})^2 \geq 0$$

thus, they both hold true. This means that their product satisfies

$$(a+b)(1+ab) \geq (2\sqrt{ab})(2\sqrt{ab}) = 4ab$$

and we're done. □

II.

$$a+b+1 \geq 2\sqrt{a+b}$$

Proof. Let $x = a+b$ so that the inequality is equivalent with

$$x+1 \geq 2\sqrt{x}$$

which is equivalent to

$$(\sqrt{x}-1)^2 \geq 0$$

so we're done. □

III.

$$(a+1)(b+1)(1+ab) \geq 8ab$$

Proof. As in problem 3 part I, we note the following simpler inequalities

$$a+1 \geq 2\sqrt{a} \quad (1.3)$$
$$b+1 \geq 2\sqrt{b} \quad (1.4)$$
$$1+ab \geq 2\sqrt{ab} \quad (1.5)$$

All three inequalities follow from the fact that for any non-negative real number x we have

$$x+1 \geq 2\sqrt{x} \iff (\sqrt{x}-1)^2 \geq 0$$

Furthermore, we have that their product satisfies

$$(a+1)(b+1)(1+ab) \geq (2\sqrt{a})(2\sqrt{b})(2\sqrt{ab}) = 8ab$$

□

IV.

$$(a^2-b^2)(a-b) \geq 0$$

Proof. By difference of squares, we have that

$$(a^2-b^2)(a-b) = (a+b)(a-b)^2$$

and we're done since both $a+b$ and $(a-b)^2$ are non-negative. □

V.

$$\frac{(a^3-b^3)(a-b)}{3} \geq ab(a-b)^2$$

Proof. As in the last problem, we note a special factorization. In this case we use difference of cubes

$$\frac{(a^3-b^3)(a-b)}{3} = \frac{(a^2+ab+b^2)(a-b)^2}{3}$$

Then note that $a^2+b^2 \geq 2ab$ so we can say that $a^2+ab+b^2 \geq 3ab$. Thus we have

$$\frac{(a^2+ab+b^2)(a-b)^2}{3} \geq \frac{3ab(a-b)^2}{3} = ab(a-b)^2$$

which is what we wanted to prove, so we're done! □

4. (Grade 8 Romanian National Math Olympiad, 2008) (Part a) Prove that for all positive reals u, v, x, y the following inequality takes place:

$$\frac{u}{x} + \frac{v}{y} \geq \frac{4(uy + vx)}{(x + y)^2}$$

Proof. We begin by taking a common denominator on the left hand side

$$\frac{u}{x} + \frac{v}{y} = \frac{uy + vx}{xy}$$

so our inequality is equivalent with

$$\frac{uy + vx}{xy} \geq \frac{4(uy + vx)}{(x + y)^2}$$

or

$$\begin{aligned} \frac{uy + vx}{xy} - \frac{4(uy + vx)}{(x + y)^2} &\geq 0 \\ (uy + vx)\left(\frac{1}{xy} - \frac{4}{(x + y)^2}\right) &\geq 0 \\ \frac{(uy + vx)((x + y)^2 - 4xy)}{xy(x + y)^2} &\geq 0 \\ \frac{(uy + vx)(x - y)^2}{xy(x + y)^2} &\geq 0 \end{aligned}$$

which clearly holds for positive reals u, v, x, y. Since the steps are reversible, we have that original inequality is solved. □

1.2 The AM-GM Inequality

The next important inequality is the AM-GM inequality, or the Arithmetic Mean - Geometric Mean inequality. In example 1.1.1, we proved the AM-GM inequality for the $n = 2$ case. Here we have its generalization.

Theorem 1.2.1 (AM-GM Inequality)**:** Let $a_1, a_2, \cdots a_n$ be non-negative real numbers, then,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Proof. (By Cauchy) We will prove inductively that the inequality satisfies for any $n = 2^k$ where k is a natural number. We'll start by proving the $n = 2$ case:

$$\begin{aligned}
\frac{a_1 + a_2}{2} &\geq \sqrt{a_1 a_2} \\
a_1 + a_2 &\geq 2\sqrt{a_1 a_2} \\
(a_1 + a_2)^2 &\geq 4a_1 a_2 \\
a_1^2 + 2a_1 a_2 + a_2^2 &\geq 4a_1 a_2 \\
a_1^2 - 2a_1 a_2 + a_2^2 &\geq 0 \\
(a_1 - a_2)^2 &\geq 0
\end{aligned}$$

Thus, the inequality follows by the Trivial Inequality, with equality when $a_1 = a_2$. Now we assume the inequality holds for some $n = 2^k$ and prove for $n = 2^{k+1}$:

$$\begin{aligned}
\frac{a_1 + a_2 + \cdots + a_{2^{k+1}}}{2^{k+1}} &= \frac{\frac{a_1 + \cdots + a_{2^k}}{2^k} + \frac{a_{2^k+1} + \cdots + a_{2^{k+1}}}{2^k}}{2} \\
\frac{a_1 + a_2 + \cdots + a_{2^{k+1}}}{2^{k+1}} &\geq \frac{\sqrt[2^k]{a_1 a_2 \cdots a_{2^k}} + \sqrt[2^k]{a_{2^k+1} \cdots a_{2^{k+1}}}}{2} \\
\frac{a_1 + a_2 + \cdots + a_{2^{k+1}}}{2^{k+1}} &\geq \sqrt[2^{k+1}]{a_1 a_2 \cdots a_{2^{k+1}}}.
\end{aligned}$$

Note that in the last step we applied the AM-GM inequality for the $n = 2$ case. So far, we have proved the AM-GM inequality for all powers of 2. To prove the inequality for all n: we take any n and let m be such that $2^m < n \leq 2^{m+1}$ (it's important to note that such m always exists) and set $p = \frac{a_1+\cdots+a_n}{n}$. Applying the AM-GM inequality for 2^{m+1} terms we have

$$\begin{aligned}
\frac{a_1 + a_2 + \cdots + a_n + (2^{m+1} - n)p}{2^{m+1}} &\geq \sqrt[2^{m+1}]{a_1 a_2 \cdots a_n p^{2^{m+1}-n}} \\
\frac{pn + (2^{m+1} - n)p}{2^{m+1}} &\geq \sqrt[2^{m+1}]{a_1 a_2 \cdots a_n p^{2^{m+1}-n}} \\
p &\geq \sqrt[2^{m+1}]{a_1 a_2 \cdots a_n p^{2^{m+1}-n}} \\
p^{2^{m+1}} &\geq a_1 a_2 \cdots a_n p^{2^{m+1}-n} \\
p^n &\geq a_1 a_2 \cdots a_n \\
p &\geq \sqrt[n]{a_1 a_2 \cdots a_n} \\
\frac{a_1 + a_2 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \cdots a_n}
\end{aligned}$$

By using previously proven cases of the AM-GM inequality, we kept the conditions for equality. Therefore, equality holds when $a_1 = a_2 = \cdots = a_n$. □

Example 1.2.2: Let a, b and c be non-negative real numbers such that $abc = 1$. Prove that

$$a + b + c \geq 3$$

Proof. The AM-GM inequality tells us that,

$$\frac{a + b + c}{3} \geq \sqrt[3]{abc}$$

By substituting $abc = 1$ and multiplying by 3 we have,

$$a + b + c \geq 3$$

which is what we wanted to prove, so we are done. □

In the next example, it is important to note that if a, b, c and d are non-negative real numbers and $a \geq b$, $c \geq d$, then $ac \geq bd$.

Example 1.2.3: Let a, b and c be positive real numbers. Prove that

$$(a+b)(b+c)(a+c) \geq 8abc$$

Proof. The AM-GM Inequality tells us that,

$$\begin{aligned} a+b &\geq 2\sqrt{ab} \\ b+c &\geq 2\sqrt{bc} \\ a+c &\geq 2\sqrt{ac} \end{aligned}$$

By multiplying these inequalities together we get,

$$(a+b)(b+c)(a+c) \geq 8abc$$

And we're done! □

Our final two examples will show how useful the AM-GM inequality can be with an olympiad level problem.

Example 1.2.4: (St. Petersburg City Mathematical Olympiad, 1999) Let $x_0 > x_1 > \cdots > x_n$ be real numbers. Prove that

$$x_0 + \frac{1}{x_0 - x_1} + \frac{1}{x_1 - x_2} + \cdots + \frac{1}{x_{n-1} - x_n} \geq x_n + 2n.$$

Proof. Let $a_k = x_k - x_{k+1} > 0$ so that our inequality is equivalent to

$$x_0 - x_n + \frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_{n-1}} \geq 2n.$$

Next, we notice that $a_0 + a_1 + \cdots + a_{n-1} = x_0 - x_n$, so our inequality is again equivalent to

$$(a_0 + a_1 + \cdots + a_{n-1}) + \frac{1}{a_0} + \cdots + + \frac{1}{a_{n-1}} \geq 2n$$

or,

$$\left(a_0 + \frac{1}{a_0}\right) + \left(a_1 + \frac{1}{a_1}\right) + \cdots + \left(a_{n-1} + \frac{1}{a_{n-1}}\right) \geq 2n.$$

Finally, by the AM-GM Inequality, we have that $a_k + \dfrac{1}{a_k} \geq 2$. Applying this inequality to each term in the previous inequality immediately gives us our result. □

Example 1.2.5: (IMO, 2012) Let $n \geq 3$ be a natural number, and let $a_2, a_3, \cdots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

Proof. Note that for all $2 \leq k \leq n$, we have

$$(1 + a_k)^k = \left(\left(\underbrace{\frac{1}{k-1} + \frac{1}{k-1} + \cdots + \frac{1}{k-1}}_{k-1} \right) + a_k \right)^k \geq \frac{k^k a_k}{(k-1)^{k-1}}$$

Multiplying all of these terms together we get

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n \geq \left(\frac{2^2 a_2}{1^1} \right) \left(\frac{3^3 a_3}{2^2} \right) \cdots \left(\frac{n^n a_n}{(n-1)^{n-1}} \right)$$

It's easy to see that all of the k^k terms cancel out except for n^n and 1^1, so we're left with

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n \geq a_2 a_3 \cdots a_n \cdot n^n$$

but we know that $a_2 a_3 \cdots a_n = 1$, so it's equivalent with

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n \geq n^n.$$

Lastly, we need to prove that the equality case never happens. Since we applied AM-GM with terms a_k and $k-1$ terms of $\frac{1}{k-1}$, equality happens when $a_k = \frac{1}{k-1}$. However, this does not satisfy the condition that $a_2 a_3 \cdots a_n = 1$. So equality can never happen, which is what we wanted to prove. □

1.2.1 Practice Problems

1. Let a and b be positive real numbers. Prove that

   I.
   $$2(a^2+b^2) \geq (a+b)^2$$

   II.
   $$\frac{a}{b}+\frac{b}{a} \geq 2$$

   III.
   $$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$$

   IV.
   $$(a+2b)(b+2a) > 8ab$$

   Why is equality not possible?

   V.
   $$a^3+b^3 \geq ab(a+b)$$

2. Let a, b and c be positive real numbers. Prove that

   I.
   $$a^3+8b^3+27c^3 \geq 18abc$$

   II.
   $$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

   III.
   $$a+b+c \geq 2\sqrt{a+b+c}-1$$

   IV.
   $$(a+b+c)\sqrt{2} \geq \sqrt[4]{2ab(a^2+b^2)}+\sqrt[4]{2bc(b^2+c^2)}+\sqrt[4]{2ac(a^2+c^2)}$$

   V. (Nesbitt's Inequality)
   $$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} \geq \frac{3}{2}$$

1.2.2 Solutions

1. Let a and b be positive real numbers. Prove that

I.
$$2(a^2+b^2) \geq (a+b)^2$$

Proof. We start by expanding both sides and simplifying

$$2a^2+2b^2 \geq a^2+2ab+b^2$$

is equivalent to
$$a^2+b^2 \geq 2ab$$

which follows from the AM-GM inequality. □

II.
$$\frac{a}{b}+\frac{b}{a} \geq 2$$

Proof. Let $x=\frac{a}{b}$ and the inequality is equivalent to

$$x+\frac{1}{x} \geq 2$$

which follows from AM-GM as we have

$$x+\frac{1}{x} \geq 2\sqrt{x \cdot \frac{1}{x}} = 2$$

□

III.
$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$$

Proof. By AM-GM we have

$$a+b \geq 2\sqrt{ab}$$

and

$$\frac{1}{a}+\frac{1}{b} \geq 2\sqrt{\frac{1}{ab}}$$

multiply these two inequalities together and you get

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq (2\sqrt{ab})\left(2\sqrt{\frac{1}{ab}}\right) = 4$$

which is what we wanted to prove. □

IV.

$$(a+2b)(b+2a) > 8ab$$

Why is equality not possible?

Proof. By AM-GM we have

$$a+2b \geq 2\sqrt{2ab}$$

with equality when $a = 2b$ and

$$b+2a \geq 2\sqrt{2ab}$$

with equality when $b = 2a$. When we multiply these inequalities together, we get

$$(a+2b)(b+2a) \geq 8ab$$

and equality only holds when $a = 2b$ and $b = 2a$ which cannot happen simultaneously unless $a = b = 0$ but a and b are positive real numbers so this case does not occur. Thus the inequality is strict. □

V.

$$a^3+b^3 \geq ab(a+b)$$

Proof. By the AM-GM inequality, we have

$$a^3 + a^3 + b^3 \geq 3\sqrt[3]{a^3 \cdot a^3 \cdot b^3} = 3a^2b$$

and

$$b^3 + b^3 + a^3 \geq 3\sqrt[3]{b^3 \cdot b^3 \cdot a^3} = 3ab^2$$

adding these inequalities together we get

$$3(a^3 + b^3) \geq 3(a^2b + ab^2) = 3ab(a + b)$$

Moreover, after dividing by 3 this inequality is equivalent to

$$a^3 + b^3 \geq ab(a + b).$$

□

2. Let a, b and c be positive real numbers. Prove that

I.

$$a^3 + 8b^3 + 27c^3 \geq 18abc$$

Proof. This a pretty straightforward application of the AM-GM inequality. We note that by the AM-GM Inequality, we have

$$a^3 + 8b^3 + 27c^3 = a^3 + (2b)^3 + (3c)^3 \geq 3\sqrt[3]{a^3 \cdot (2b)^3 \cdot (3c)^3} = 18abc.$$

□

II.

$$(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$$

Proof. We apply the same idea we used earlier and use the AM-GM Inequality on each term of the two terms. By the AM-GM Inequality, we have

$$a + b + c \geq 3\sqrt[3]{abc}$$

as well as

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3\sqrt[3]{\frac{1}{abc}}$$

next we multiply these two inequalities together to get

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq (3\sqrt[3]{abc})\left(3\sqrt[3]{\frac{1}{abc}}\right) = 9$$

which is what we wanted to prove. □

III.

$$a+b+c \geq 2\sqrt{a+b+c} - 1$$

Proof. Let $x = a+b+c$ then the inequality is equivalent with

$$x \geq 2\sqrt{x} - 1$$

or

$$x + 1 \geq 2\sqrt{x}$$

which follows directly from the AM-GM Inequality. □

IV.

$$(a+b+c)\sqrt{2} \geq \sqrt[4]{2ab(a^2+b^2)} + \sqrt[4]{2bc(b^2+c^2)} + \sqrt[4]{2ac(a^2+c^2)}$$

Proof. This problem is indeed a bit tricky. We note that by the AM-GM Inequality we have

$$\sqrt{2ab(a^2+b^2)} \leq \frac{(2ab)+(a^2+b^2)}{2} = \frac{(a+b)^2}{2}$$

Furthermore, if we take the square root of both sides we get

$$\sqrt[4]{2ab(a^2+b^2)} \leq \frac{a+b}{\sqrt{2}}$$

By adding this inequality cyclically, we get

$$\frac{2(a+b+c)}{\sqrt{2}} \geq \sqrt[4]{2ab(a^2+b^2)} + \sqrt[4]{2bc(b^2+c^2)} + \sqrt[4]{2ac(a^2+c^2)}$$

and it's clear that

$$\frac{2(a+b+c)}{\sqrt{2}} = (a+b+c)\sqrt{2}$$

so we're done! □

V. (Nesbitt's Inequality)

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

Proof. Rewrite the inequality with cyclic notation so that it is equivalent with

$$\sum_{cyc} \frac{a}{b+c} \geq \frac{3}{2}$$

Furthermore, we may add 1 to each term so that they share the same denominator

$$\sum_{cyc} \left(\frac{a+b+c}{b+c}\right) = \sum_{cyc} \left(\frac{a}{b+c} + 1\right) \geq \frac{3}{2} + 3 = \frac{9}{2}$$

or

$$(a+b+c)\left(\sum_{cyc} \frac{1}{b+c}\right) \geq \frac{9}{2}$$

moreover, if we let $x = b+c$, $y = a+c$ and $z = a+b$, then our inequality is equivalent with

$$\left(\frac{x+y+z}{2}\right)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq \frac{9}{2}$$

which, after multiplying by 2, gives us

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9$$

but this is equivalent to problem 2 part II, we simply apply the same ideas and we're done.

Remarks:

$$\sum_{cyc} \frac{a}{b+c} = \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}$$

$$\sum_{cyc} \left(\frac{a}{b+c}+1\right) = \left(\frac{a}{b+c}+1\right) + \left(\frac{b}{a+c}+1\right) + \left(\frac{c}{a+b}+1\right)$$

□

1.3 The Cauchy-Schwarz Inequality

In this section, we'll present a powerful theorem, follow it with some examples and end off with a nice set of problems.

Theorem 1.3.1 (The Cauchy-Schwarz Inequality)**:** Let $a_1, a_2, \cdots a_n, b_1, b_2, \cdots, b_n$ be real numbers, then,

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$$

with equality if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

Proof. Let $f_i(x) = (a_i x - b_i)^2$ and consider the sum

$$P(x) = \sum_{i=1}^{n} f_i(x) = (a_1^2 + \cdots + a_n^2)x^2 - 2x(a_1 b_1 + \cdots + a_n b_n) + (b_1^2 + \cdots + b_n^2)$$

It's clear that since $P(x)$ is the sum of squares then it is always non-negative, so $P(x) \geq 0$. Equality happens when $f_1(x) = \cdots = f_n(x) = 0$ or

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$

Furthermore, $P(x)$'s discriminant should therefore be non-positive (as the roots must be complex or 0). And so we have

$$\begin{aligned}
(-2(a_1 b_1 + \cdots + a_n b_n))^2 - 4(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) &\leq 0 \\
4(a_1 b_1 + \cdots + a_n b_n)^2 - 4(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) &\leq 0 \\
(a_1 b_1 + \cdots + a_n b_n)^2 - (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) &\leq 0.
\end{aligned}$$

Which is equivalent to our original inequality. □

Example 1.3.2: Let a, b, c be real numbers. Prove that

$$a^2 + b^2 + c^2 \geq ab + bc + ac$$

Proof. By the Cauchy-Schwarz Inequality, we have that,

$$(a^2+b^2+c^2)(b^2+c^2+a^2) \geq (ab+bc+ac)^2$$

Note that this is equivalent to

$$(a^2+b^2+c^2)^2 \geq (ab+bc+ac)^2$$

And the result is evident, so we are done. (Note: We solved this problem using perfect squares in the previous section. This shows us that inequalities can have multiple solutions, and in fact, most inequalities do.) □

Example 1.3.3: Let a, b and c be positive real numbers. Prove that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

Proof. Since a, b and c are positive real numbers we can let a, b and c be x^2, y^2 and z^2, respectively. This makes our inequality now equivalent with,

$$(x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) \geq 9$$

Next, we note that by the Cauchy-Schwarz Inequality we have,

$$(x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) \geq \left(x\cdot\frac{1}{x}+y\cdot\frac{1}{y}+z\cdot\frac{1}{z}\right)^2 = 9$$

and we're done. □

Example 1.3.4: (Ireland, 1998) Prove that if a, b, c are positive real numbers, then,

$$\frac{9}{a+b+c} \leq 2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

and,

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \leq \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proof. We'll prove the second inequality. By applying the Cauchy-Schwarz Inequality on two variables as we did in the previous example, which had three variables, we have,

$$\begin{aligned} (a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4 \implies \frac{1}{a}+\frac{1}{b} &\geq \frac{4}{a+b} \\ (b+c)\left(\frac{1}{b}+\frac{1}{c}\right) \geq 4 \implies \frac{1}{b}+\frac{1}{c} &\geq \frac{4}{b+c} \\ (a+c)\left(\frac{1}{a}+\frac{1}{c}\right) \geq 4 \implies \frac{1}{a}+\frac{1}{c} &\geq \frac{4}{a+c} \end{aligned}$$

Adding these inequalities together we get,

$$2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 4\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{a+c}\right)$$

Next, we divide by 4 and we're done. The first inequality is left as an exercise for the reader. □

Example 1.3.5: Let a, b, c, x, y and z be positive real numbers such that $a+b+c = x+y+z$. Prove that

$$\frac{a^2}{y+z}+\frac{b^2}{x+z}+\frac{c^2}{x+y} \geq \frac{a+b+c}{2}$$

Proof. We start by noting that the only variables used in the right hand side of our inequality are a, b and c, hence, we want to apply the Cauchy-Schwarz Inequality in such a way that the x, y and z's are eliminated. This hints us to think of applying the Cauchy-Schwarz Inequality like so

$$((y+z)+(x+z)+(x+y))\left(\frac{a^2}{y+z}+\frac{b^2}{x+z}+\frac{c^2}{x+y}\right) \geq (a+b+c)^2$$

Next, we note that $(y+z)+(x+z)+(x+y) = 2(a+b+c)$, thus

$$\frac{a^2}{y+z}+\frac{b^2}{x+z}+\frac{c^2}{x+y} \geq \frac{a+b+c}{2}$$

□

Example 1.3.6: Let x, y and z be positive real numbers. Prove that

$$\sqrt{(x+y)(y+z)(x+z)+xyz} \geq x\sqrt{\frac{y+z}{2}} + y\sqrt{\frac{x+z}{2}} + z\sqrt{\frac{x+y}{2}}$$

Proof. We use the identity shown in the first section,

$$xyz = (x+y+z)(xy+yz+xz) - (x+y)(y+z)(x+z)$$

to infer that,

$$(x+y)(y+z)(x+z) + xyz = (x+y+z)(xy+yz+xz).$$

We can deduce that our inequality is equivalent to proving

$$\sqrt{(x+y+z)(xy+yz+xz)} \geq x\sqrt{\frac{y+z}{2}} + y\sqrt{\frac{x+z}{2}} + z\sqrt{\frac{x+y}{2}}.$$

or

$$\sqrt{2(x+y+z)(xy+yz+xz)} \geq x\sqrt{y+z} + y\sqrt{x+z} + z\sqrt{x+y}$$

This follows from the Cauchy-Schwarz Inequality as we know that

$$2(xy+yz+xz) = x(y+z) + y(x+z) + z(x+y)$$

and we're done. □

1.3.1 Practice Problems

When solving these problems you need only to remember one thing: be clever!

1. (Ireland, 1999) Let a, b, c, d be positive real numbers which sum up to 1. Prove that
$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2}$$

2. Let a, b and c be real numbers. Prove that
$$2a^2 + 3b^2 + 6c^2 \geq (a+b+c)^2$$

3. Let a, b and c be positve real numbers. Prove that

$$\frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b} \geq a + b + c$$

4. (Central American and Caribbean Math Olympiad, 2009) Let x, y, z be real numbers such that $xyz = 1$. Prove that

$$(x^2+1)(y^2+1)(z^2+1) \geq \left(1 + \frac{x}{y}\right)\left(1 + \frac{y}{z}\right)\left(1 + \frac{z}{x}\right)$$

When is there equality?

5. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$a^2 + b^2 + c^2 \geq a + b + c$$

6. (Puerto Rican Mathematical Olympiad Ibero TST, 2009) Let h_a, h_b and h_c be the altitudes of triangle ABC and let r be its inradius. Prove that

$$h_a + h_b + h_c \geq 9r$$

7. (Czech and Slovak Republics, 1999) For arbitrary positive numbers a, b and c, Prove that

$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \geq 1$$

Note: This is also a problem from the International Zhautykov Olympiad in 2005.

8. (Iran, 1998) Let $x, y, z > 1$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$. Prove that

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

9. (Belarus IMO TST, 1999) Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ac} \geq \frac{3}{2}$$

10. (France IMO TST, 2006) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)} \geq \frac{3}{4}.$$

When is there equality?

1.3.2 Solutions

1. (Ireland, 1999) Let a, b, c, d be positive real numbers which sum up to 1. Prove that
$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\geq\frac{1}{2}$$

Proof. By the Cauchy-Schwarz Inequality, we have
$$\left(\sum_{cyc} a+b\right)\left(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\right)\geq(a+b+c+d)^2$$
or
$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\geq\frac{(a+b+c+d)^2}{2(a+b+c+d)}.$$
Since $a+b+c+d=1$, we have that
$$\frac{(a+b+c+d)^2}{2(a+b+c+d)}=\frac{1}{2}.$$
So we have
$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\geq\frac{1}{2}$$
which is what we wanted to prove. □

2. Let a, b and c be real numbers. Prove that
$$2a^2+3b^2+6c^2\geq(a+b+c)^2$$

Proof. We start by noting the identity
$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$$
also, by the Cauchy-Schwarz Inequality, we have
$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)(2a^2+3b^2+6c^2)\geq(a+b+c)^2.$$
Using our identity the result is evident! □

3. Let a, b and c be positve real numbers. Prove that

$$\frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b} \geq a + b + c$$

Proof. By the Cauchy-Schwartz Inequality, we have

$$(c + a + b)\left(\frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b}\right) \geq (a + b + c)^2$$

then divide both sides by $a + b + c$ and we get the desired result. □

4. (Central American and Caribbean Math Olympiad, 2009) Let x, y, z be real numbers such that $xyz = 1$. Prove that

$$(x^2 + 1)(y^2 + 1)(z^2 + 1) \geq \left(1 + \frac{x}{y}\right)\left(1 + \frac{y}{z}\right)\left(1 + \frac{z}{x}\right)$$

When is there equality?

Proof. To simplify the inequality, we multiply the right hand side by xyz

$$(x^2 + 1)(y^2 + 1)(z^2 + 1) \geq \left(1 + \frac{x}{y}\right)\left(1 + \frac{y}{z}\right)\left(1 + \frac{z}{x}\right)xyz$$

and note that

$$\left(1 + \frac{x}{y}\right)\left(1 + \frac{y}{z}\right)\left(1 + \frac{z}{x}\right)xyz = (x + y)(y + z)(z + x)$$

Furthermore, by the Cauchy-Schwarz Inequality, we have

$$(x^2 + 1)(1 + y^2) \geq (x + y)^2 \implies \sqrt{(x^2 + 1)(y^2 + 1)} \geq x + y$$

by multiplying this inequality cyclically we get the desired result and we're done. There is equality when $x = y = z = 1$. □

5. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$a^2 + b^2 + c^2 \geq a + b + c$$

Proof. First we note that by the AM-GM inequality we have

$$a^2 + b^2 + c^2 \geq 3\sqrt[3]{a^2b^2c^2} = 3$$

next we multiply by $a^2 + b^2 + c^2$ on both sides

$$(a^2 + b^2 + c^2)^2 \geq 3(a^2 + b^2 + c^2)$$

but by the Cauchy-Schwarz Inequality we have

$$3(a^2 + b^2 + c^2) \geq (a + b + c)^2$$

thus

$$(a^2 + b^2 + c^2)^2 \geq (a + b + c)^2$$

then since both terms are positive we can take the square root and we get

$$a^2 + b^2 + c^2 \geq a + b + c$$

which is what we wanted to prove, so we're done. □

6. (Puerto Rican Mathematical Olympiad Ibero TST, 2009) Let h_a, h_b and h_c be the altitudes of triangle ABC and let r be its inradius. Prove that

$$h_a + h_b + h_c \geq 9r$$

Proof. We'll start by proving the geometric identity

$$\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}.$$

It follows from the fact that

$$[ABC] = \frac{ah_a}{2} = \frac{bh_b}{2} = \frac{ch_c}{2} = rs$$

where $[ABC]$ is the area of triangle ABC and a, b, c, s are the sides and semi-perimeter of the triangle, respectively. This allows us to note that

$$\frac{1}{h_a} = \frac{a}{2rs}$$

so that

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \left(\frac{1}{2rs}\right)(a+b+c)$$

but $a+b+c = 2s$ by definition so

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$$

Furthermore, by the Cauchy-Schwarz Inequality, we have

$$\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)(h_a + h_b + h_c) \geq 9$$

or

$$\left(\frac{1}{r}\right)(h_a + h_b + h_c) \geq 9.$$

Then we multiply by r and we're done! □

7. (Czech and Slovak Republics, 1999) For arbitrary positive numbers a, b and c, Prove that

$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \geq 1$$

Note: This is also a problem from the International Zhautykov Olympiad in 2005.

Proof. By the Cauchy-Schwarz Inequality, it is clear that

$$\left(\sum_{cyc} a(b+2c)\right)\left(\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b}\right) \geq (a+b+c)^2$$

Therefore, the following inequality also holds

$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \geq \frac{(a+b+c)^2}{a(b+2c)+b(c+2a)+c(a+2b)}$$

and finally, since

$$a(b+2c)+b(c+2a)+c(a+2b) = (a+b+c)^2$$

the right hand side is equal to 1 and we're done. □

8. (Iran, 1998) Let $x, y, z > 1$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$. Prove that

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

Proof. We start by rewriting the condition

$$\begin{aligned} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= 2 \\ -\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) &= -2 \\ 3 - \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) &= 1 \\ \frac{x-1}{x} + \frac{y-1}{y} + \frac{z-1}{z} &= 1. \end{aligned}$$

Next, by the Cauchy-Schwarz Inequality, we have

$$\sqrt{(x+y+z)\left(\sum_{cyc} \frac{x-1}{x}\right)} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

or

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

which is what we wanted to prove, so we're done! □

9. (Belarus IMO TST, 1999) Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ac} \geq \frac{3}{2}$$

Proof. By the Cauchy-Schwarz Inequality, we have

$$((1+ab) + (1+bc) + (1+ac))\left(\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ac}\right) \geq 9$$

or

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac}\ge\frac{9}{(1+ab)+(1+bc)+(1+ac)}$$

thus, it is sufficient to prove that

$$\frac{9}{(1+ab)+(1+bc)+(1+ac)}\ge\frac{3}{2}$$

or, after simplifying,

$$3\ge ab+bc+ac$$

which, using the condition, is equivalent to

$$a^2+b^2+c^2\ge ab+bc+ac$$

or

$$(a-b)^2+(b-c)^2+(c-a)^2\ge 0$$

which of course is true. □

10. (France IMO TST, 2006) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a}{(a+1)(b+1)}+\frac{b}{(b+1)(c+1)}+\frac{c}{(c+1)(a+1)}\ge\frac{3}{4}.$$

When is there equality?

Proof. We begin by multiplying both sides by $4(a+1)(b+1)(c+1)$

$$4(a(c+1)+b(a+1)+c(b+1))\ge 3(a+1)(b+1)(c+1)$$

which is equivalent to

$$4(ab+bc+ac+a+b+c)\ge 3(abc+ab+bc+ac+a+b+c+1)$$

or

$$ab+bc+ac+a+b+c\ge 3(abc+1)=3(1+1)=6$$

which follows from AM-GM and we're done. We have equality when $a = b = c = 1$. Note that we didn't use the Cauchy-Schwarz Inequality. See if you can find a Cauchy-Schwarz solution. □

1.4 Using Inequalities to Solve Optimization Problems

"When, with a fixed perimeter, is a rectangle's area the greatest?" "Given two positive numbers with a fixed product P, what is the smallest possible value of their sum?" These are the types of problems that we'll tackle in this section. They ask to find the conditions for maximums and minimums. So let's start with the first example.

Example 1.4.1: When, with a fixed perimeter, is a rectangle's area the greatest?

Proof. Let the fixed perimeter be S and sides of the rectangle be a and b, so that $S = 2a + 2b$. Now note that $a - \frac{S}{4} = \frac{S}{4} - b$ follows from $S = 2a + 2b$ and so we can let $k = a - \frac{S}{4} = \frac{S}{4} - b$. This, in turn, tells us that a and b can be written as $\frac{S}{4} + k$ and $\frac{S}{4} - k$, respectively. Thus, we have

$$\text{Area of rectangle} = ab = \left(\frac{S}{4} + k\right)\left(\frac{S}{4} - k\right) = \frac{S^2}{16} - k^2$$

Since S is fixed constant, we have that the only variable affecting the area of the rectangle is k. By minimizing k^2 we maximize the area. Since $k^2 \geq 0$, the lowest possible value for k^2 is then 0, which happens when $k = 0$. When $k = 0$ we have that $a = b = \frac{S}{4}$ and so the area of rectangle with fixed perimeter is maximized when it is a square. Alternatively, by the AM-GM inequality we have that

$$\frac{a+b}{2} \geq \sqrt{ab} \Leftrightarrow ab \leq \left(\frac{a+b}{2}\right)^2 = \frac{S^2}{16}$$

with equality if and only if $a = b$, i.e. when the rectangle is a square. □

Note that we were able to find the maximum area by rewriting the expression in terms of only one variable. When working on optimization problems, it's important to try to make the expression dependent on less variables.

Example 1.4.2: Given two positive numbers with a fixed product P, what is the smallest possible value of their sum?

Proof. Let a and b be the two numbers, and so $ab = P$. By the AM-GM Inequality, we have

$$\begin{aligned}\frac{a+b}{2} &\geq \sqrt{ab} \\ a+b &\geq 2\sqrt{ab} \\ a+b &\geq \boxed{2\sqrt{P}}\end{aligned}$$

and so the smallest possible value for their sum is $\boxed{2\sqrt{P}}$ and it is achieved when $a = b = \sqrt{P}$. □

Example 1.4.3: We are given a segment AB of length L. Consider any point P on this segment and rotate PA around the point P so that A is taken to A' and $A'PB$ is a right triangle ($\angle A'PB = 90°$). What is the shortest possible distance between A' and B in terms of L and where is this point?

Proof. Note that if we let $PA = x$ and $PB = y$ then $x+y = L$ and the distance between $A'B$ is $\sqrt{PA^2 + PB^2} = \sqrt{x^2+y^2}$ (by Pythagoras and $PA' = PA$). So we wish to minimize $\sqrt{x^2+y^2}$. Applying the same idea as before we can note that there exists a $k \in \mathbb{R}$ such that $x = \frac{L}{2} + k$ and $y = \frac{L}{2} - k$. Thus, we wish to minimize

$$\sqrt{x^2+y^2} = \sqrt{\left(\frac{L}{2}+k\right)^2 + \left(\frac{L}{2}-k\right)^2} = \sqrt{\frac{L^2}{2} + 2k^2}$$

But again, $k^2 \geq 0$ and L is fixed so the minimum is $\sqrt{\frac{L^2}{2}}$ or $\frac{\sqrt{2}L}{2}$. Since the minimum is when $k = 0$, we get that $PA = PB$ and so P is the midpoint of L. Alternatively, we can note that

$$\begin{aligned}(x-y)^2 &\geq 0 \\ x^2+y^2 &\geq 2xy \\ 2(x^2+y^2) &\geq (x+y)^2 \\ \sqrt{x^2+y^2} &\geq \frac{x+y}{\sqrt{2}}\end{aligned}$$

$$\sqrt{x^2+y^2} \geq \frac{\sqrt{2}L}{2}$$

Thus, the minimum is $\frac{\sqrt{2}L}{2}$ and it is achieved when $(x-y)^2 = 0$ which implies that $x = y$ (i.e. P is the midpoint of L). □

Example 1.4.4: Determine the largest and smallest possible value of $\sin x + \cos x$ where $x \in \mathbb{R}$.

Proof. Here we'll exploit a nice identity about the sine function that states that

$$\sin\left(x+\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}\right)(\sin x + \cos x)$$

for all $x \in \mathbb{R}$. Since $-1 \leq \sin\left(x+\frac{\pi}{4}\right) \leq 1$ by definition, it follows that

$$-\sqrt{2} \leq \sin x + \cos x \leq \sqrt{2}$$

We must also show that these max and min values are attainable. To do this we must find what values of x make $\sin\left(x+\frac{\pi}{4}\right)$ equal to its extremes (1 and -1). We can quickly show that $x = \frac{\pi}{4}$ makes the expression equal to 1 and that $x = \frac{5\pi}{4}$ results in -1. So the max and min values are indeed attainable.

Here's another way one might solve this problem. Let $a = \sin x$ and $b = \cos x$ and note that $a^2 + b^2 = 1$. We now look for an inequality relating $a^2 + b^2$ and $a + b$.

$$\begin{aligned} (a-b)^2 &\geq 0 \\ a^2+b^2 &\geq 2ab \\ 2(a^2+b^2) &\geq (a+b)^2 \\ 2 &\geq (a+b)^2. \end{aligned}$$

Therefore, $|a+b| \leq \sqrt{2}$ which is equivalent to

$$-\sqrt{2} \leq \sin x + \cos x \leq \sqrt{2}.$$

□

The next example presents an olympiad level optimization problem and the kind of solution you are expected to present at these competitions.

Example 1.4.5: (Ibero, 2010) The arithmetic, geometric and harmonic mean of two distinct positive integers are different integers. Find the smallest possible value for the arithmetic mean.

Proof. (Posted by uglysolutions at artofproblemsolving.com)[1] Let our positive integers be $a \neq b$. Let $d = \gcd(a, b)$, thus $a = dm$, $b = dn$, with $\gcd(m, n) = 1$.

The harmonic mean $\frac{2}{\frac{1}{a}+\frac{1}{b}}$ is an integer, therefore $a + b \mid 2ab$, which means $d(m + n) \mid 2d^2mn$ and so $m + n \mid 2dmn$ which leaves us with $m + n \mid 2d$ since $m + n$ and mn are relatively prime.

The arithmetic mean is an integer, thus $2 \mid d(m + n)$.

The geometric mean is an integer, hence d^2mn is a perfect square which implies that mn is a perfect square and so both m and n are perfect squares, because they are relatively prime.

We want to find the minimum value of $\frac{d(m+n)}{2}$ under these conditions.

If $m + n$ is odd, we get $m + n \mid d$ and $2 \mid d$, thus $d \geq 2(m + n)$. But $m+n \geq 1+4 = 5$, thus $\frac{d(m+n)}{2} \geq \frac{10\times 5}{2} = 25$. If $m+n$ is even, we get $d \geq \frac{m+n}{2}$, since $m + n \geq 1 + 9 = 10$, again we get $\frac{d(m+n)}{2} \geq \frac{5\times 10}{2} = 25$.

So the answer is 25. Examples attaining this minimum value are $(5, 45)$, $(10, 40)$. □

Example 1.4.6: Let ABC be a triangle. Suppose we keep points B and C fixed but we let A vary such that the perimeter, $2s$, stays constant. What is the largest possible area in terms of s and BC?

[1] http://www.artofproblemsolving.com/Forum/viewtopic.php?p=2029531

Proof. We let $a = BC$ and note that the points that satisfy the conditions define an ellipse whose foci are B and C and the point A lies on the graph of the ellipse. Furthermore, we maximize the area by maximizing the distance from the point A to the segment BC (altitude in the triangle) since the base remains constant. Clearly, the point with the largest distance is the intersection of the perpendicular bisector of BC with the graph of the parabola. This point is then equidistant from the foci, so our triangle is isosceles. If we let the second/third side be denoted as b, then the area of the triangle is

$$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s(s-a)(s-b)^2} = \frac{a}{2}\sqrt{s(s-a)}$$

which follows from noting that

$$a + 2b = 2s \implies s - b = \frac{a}{2}.$$

□

1.4.1 Practice Problems

1. Let a, b, c be positive real numbers such that $a + b + c = 3$. Determine the maximum attainable value of $ab + bc + ac$.

2. Determine the least surface area that a rectangular box with volume 8 can have.

3. Show that of all rectangles inscribed in a given circle the square has the largest area.

4. Let P be a point inside a given triangle ABC. Let AP, BP and CP intersect sides BC, AC and AB at points D, E and F, respectively. Determine the points P that minimize the sum

$$\frac{AF}{FB} + \frac{BD}{DC} + \frac{CE}{EA}$$

5. Let a, b and c be positive real numbers such that $abc = 1$. Maximize the expression

$$P = (\max\{a, b, c\})^2 - \max\{a^2 - bc, b^2 - ac, c^2 - ab\}$$

6. (APMO, 1990) Consider all the triangles ABC which have a fixed base BC and whose altitude from A is a constant h. For which of these triangles is the product of its altitudes a maximum?

7. (IMO, 1981) Consider a variable point P inside a given triangle ABC. Let D, E, F be the feet of the perpendiculars from point P to the lines BC, CA, AB, respectively. Find all points P which minimize the sum

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}.$$

1.4.2 Solutions

1. Let a, b, c be positive real numbers such that $a+b+c=3$. Determine the maximum attainable value of $ab+bc+ac$.

   Proof. Note that

   $$(a+b+c)^2 \geq 3(ab+bc+ac) \Leftrightarrow (a-b)^2+(b-c)^2+(c-a)^2 \geq 0$$

   thus the inequality holds. Furthermore, we know that $a+b+c=3$ so our inequality is equivalent with

   $$9 \geq 3(ab+bc+ac) \implies 3 \geq ab+bc+ac.$$

   Thus, the maximum value $ab+bc+ac$ can have is 3 and it's attainable with $a=b=c=1$. □

2. Determine the least surface area that a rectangular box with volume 8 can have.

   Proof. Let the dimensions of the rectangular box be $a \times b \times c$. We have that $abc=8$ and the surface area is equal to $2(ab+bc+ac)$. The AM-GM inequality gives us that

   $$2(ab+bc+ac) \geq 2(3\sqrt[3]{(abc)^2}) = 6\sqrt[3]{64} = \boxed{24}.$$

   It's achieved when $a=b=c=2$. □

3. Show that of all rectangles inscribed in a given circle the square has the largest area.

   Proof. Assume we have an arbitrary inscribed rectangle drawn. Note that this rectangle is made up of two congruent right triangles. Thus, we need only to maximize one of the right triangles. Now consider one of the right triangles inscribed in the circle, let's call it ABC, its hypotenuse, BC, is a diameter of the circle. Its area is base times height divided by 2. Let the base be BC, the base is thus constant. So we need to maximize the height which is the distance from A to the hypotenuse (diameter). It's easy to see that this height is maximized when A is equidistant from B and C thus making ABC isosceles. If the area of ABC is maximized when it's isosceles then the area of the rectangle is maximized when it's a square and we're done.

Alternatively, we can note that the diameter (hypotenuse) is given and so if the other two sides of the right triangle are a and b then our problem turns into maximizing ab where $a^2 + b^2$ is some constant k. This follows from AM-GM as we have

$$\frac{k}{4} = \frac{a^2 + b^2}{4} \geq \frac{ab}{2}.$$

This implies that the maximum area of the triangle is $\frac{k}{4}$ with equality when $a = b$ (i.e. when the triangle is isosceles). □

4. Let P be a point inside a given triangle ABC. Let AP, BP and CP intersect sides BC, AC and AB at points D, E and F, respectively. Determine the points P that minimize the sum

$$\frac{AF}{FB} + \frac{BD}{DC} + \frac{CE}{EA}$$

Proof. Ceva's Theorem tells us that

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Thus, by AM-GM we have

$$\frac{AF}{FB} + \frac{BD}{DC} + \frac{CE}{EA} \geq 3\sqrt[3]{\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA}} = 3$$

with equality when

$$\frac{AF}{FB} = \frac{BD}{DC} = \frac{CE}{EA} = k$$

We know that $k^3 = 1$ and so $k = 1$ which implies F, D and E are midpoints. Thus, P is the centroid. □

5. Let a, b and c be positive real numbers such that $abc = 1$. Maximize the expression

$$P = (\max\{a, b, c\})^2 - \max\{a^2 - bc, b^2 - ac, c^2 - ab\}$$

Proof. Without loss of generality let $a = \max\{a, b, c\}$. Note that

$$a^2 - bc \geq b^2 - ac \Leftrightarrow (a + b + c)(a - b) \geq 0$$

since $a + b + c > 0$ and $a \geq b$ we have that $a^2 - bc \geq b^2 - ac$. Similarly, $a^2 - bc \geq c^2 - ab$. Thus, we have that

$$a^2 - bc = \max\{a^2 - bc, b^2 - ac, c^2 - ab\}.$$

This implies that

$$P = (a^2) - (a^2 - bc) = bc$$

So we need to maximize bc. Since $abc = 1$ and $b, c \leq a$ we have that $a^3 \geq abc = 1 \implies a \geq 1$ or $1 \geq \frac{1}{a} = bc$. So the maximum of bc is 1 and it's attained when $a = b = c = 1$. □

6. (APMO, 1990) Consider all the triangles ABC which have a fixed base BC and whose altitude from A is a constant h. For which of these triangles is the product of its altitudes a maximum?

Proof. Let the altitudes be h_a, h_b, h_c. Note that the conditions imply that the area remains constant and that A moves along a line parallel to BC passing through the original point A. Thus, we have

$$\frac{h_a \cdot a}{2} = \frac{h_b \cdot b}{2} = \frac{h_c \cdot c}{2} = [ABC]$$

Multiplying these together we get

$$\frac{h_a h_b h_c (abc)}{8} = [ABC]^3$$

or

$$h_a h_b h_c = \frac{8[ABC]^3}{abc}$$

which is maximized when abc is minimized (since $8[ABC]^3$ is constant). Note that a is constant as well so we only need to minimize bc. Noting that

$$\frac{bc\sin A}{2} = [ABC] \implies bc = \frac{2[ABC]}{\sin A}$$

tells us that bc is minimized when $\sin A$ is maximized. When $h_a \leq \frac{a}{2}$, we draw a circle with diameter BC and note where the line parallel to BC which passes through A intersects this circle. Let these points be P and P'. It's clear that $\angle BPC$ and $\angle BP'C$ are both $90°$ and so we can maximize $\sin A$ with these points, since $\sin P = \sin P' = 1$. When $h_a > \frac{a}{2}$, we still need to maximize $\sin A$ but in this case $\sin A = 1$ is unreachable. For this case, we know that A has to be move parallel to BC to a point where $\sin BAC$ is maximized. We let T be any point on this parallel line. By the Extended Law of Sines we have

$$\sin\angle BTC = \frac{BC}{2R}.$$

In this case R is the circumradius of triangle BTC and BC is constant. Hence, to maximize $\sin\angle BTC$ we need to minimize the circumradius R. We can minimize the circumradius by minimizing the distance from T to both B and C. This happens when T is equidistant from B and C, in other words where triangle BTC is isosceles. We can construct T by noting where BC's perpendicular bisector intersects the line passing through A that is parallel to BC. In conclusion, for $h_a \leq \frac{a}{2}$ we choose a new point A' which makes $\angle BA'C = 90°$ such that AA' is parallel to BC. For $h_a > \frac{a}{2}$, we choose the point A' which makes the triangle isosceles and maintains the same altitude. Both can be constructed as shown in the proof. □

7. (IMO, 1981) Consider a variable point P inside a given triangle ABC. Let D, E, F be the feet of the perpendiculars from point P to the lines BC, CA, AB, respectively. Find all points P which minimize the sum

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}.$$

Proof. Let BC, AC and AB be denoted as a, b, c, respectively. Similarly, denote PD, PE and PF by x, y, z, respectively. Note that we wish to minimize

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z}.$$

We know however that

$$\frac{ax}{2} = [BPC],\ \frac{by}{2} = [CPA],\ \frac{cz}{2} = [APB]$$

adding these together we get

$$\frac{ax + by + cz}{2} = [BPC] + [CPA] + [APB] = [ABC]$$

Furthermore, by the Cauchy-Schwarz Inequality, we have

$$(ax + by + cz)\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right) \geq (a + b + c)^2$$

or

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} \geq \frac{(a + b + c)^2}{ax + by + cz} = \frac{(a + b + c)^2}{2[ABC]}.$$

Which has equality when

$$\frac{ax}{\frac{a}{x}} = \frac{by}{\frac{b}{y}} = \frac{cz}{\frac{c}{z}}$$

or its equivalent

$$x^2 = y^2 = z^2 \implies x = y = z$$

Thus, P is equidistant from sides BC, AC and AB from where it's clear that P is the incenter. □

Chapter 2

Advanced Theorems and Other Methods

2.1 The Cauchy-Schwarz Inequality (Generalized)

Let's recall the Cauchy-Schwarz Inequality:

Theorem 2.1.1 (The Cauchy-Schwarz Inequality)**:** Let $a_1, a_2, \cdots a_n, b_1, b_2, \cdots, b_n$ be real numbers, then,

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$$

with equality if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

We note that in the Cauchy-Schwarz Inequality, the left hand side has two products where the terms inside are elevated to the second power. In Hölder's Inequality, we take that two and generalize it. For example, by Hölder's Inequality on positive real numbers $a_1, a_2, a_3, b_1, b_2\ b_3, c_1, c_2, c_3$, we have,

$$(a_1^3 + a_2^3 + a_3^3)(b_1^3 + b_2^3 + b_3^3)(c_1^3 + c_2^3 + c_3^3) \geq (a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3)^3$$

It's important to note that now, instead of there being two products with terms inside being elevated to the second power, there are three products with terms inside elevated to the third power. Similarly, if we were to have four products, then the terms inside would be elevated to the fourth power, and so on. Formally, this inequality is equivalent to:

Theorem 2.1.2 (Hölder's Inequality)**:** For all $a_{i_j} > 0$ where $1 \leq i \leq m$, $1 \leq j \leq n$ we have

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} a_{i_j}^{m}\right) \geq \left(\sum_{j=1}^{n}\left(\prod_{i=1}^{m} a_{i_j}\right)\right)^{m}.$$

It's derived from its more general version: Given real numbers $x_1, x_2, \cdots x_n$ and $y_1, y_2, \cdots, y_n$ we have

$$\sum_{k=1} |x_k y_k| \leq \left(\sum_{k=1} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1} |y_k|^q\right)^{\frac{1}{q}}.$$

A proof of this theorem in its vector form can be found online at `http://www.proofwiki.org/wiki/H%C3%B6lder's_Inequality_for_Sums`. Note that the Cauchy-Schwarz Inequality is Hölder's Inequality for the case $m = 2$.

Example 2.1.3: Let a, b and c be positive real numbers. Prove that

$$(a^3+2)(b^3+2)(c^3+2) \geq (a+b+c)^3$$

Proof. By Hölder's Inequality, we have that,

$$(a^3+1+1)(1+b^3+1)(1+1+c^3) \geq \left(\sqrt[3]{a^3\cdot 1\cdot 1} + \sqrt[3]{1\cdot b^3\cdot 1} + \sqrt[3]{1\cdot 1\cdot c^3}\right)^3$$

or

$$(a^3+2)(b^3+2)(c^3+2) \geq (a+b+c)^3$$

And we're done! □

Example 2.1.4: Prove the Arithmetic Mean - Geometric Mean Inequality.

Proof. The Arithmetic Mean - Geometric Mean Inequality states that for positive real numbers $a_1, a_2, \cdots, a_n$ the following inequality holds,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 a_3 \cdots a_n}$$

Hence, it is equivalent to proving that,

$$a_1 + a_2 + \cdots + a_n \geq n \sqrt[n]{a_1 a_2 a_3 \cdots a_n}$$

or

$$(a_1 + a_2 + \cdots + a_n)^n \geq (n \sqrt[n]{a_1 a_2 \cdots a_n})^n$$

Next we note that,

$$(a_1 + a_2 + \cdots + a_n)^n = \left(\sum_{cyc} a_1\right)\left(\sum_{cyc} a_2\right) \cdots \left(\sum_{cyc} a_n\right)$$

The result then follows directly by applying Hölder's Inequality, and so we are done! □

Example 2.1.5 (Junior Balkan MO, 2002)**:** Prove that for all positive real numbers a, b, c, the following inequality takes place

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}$$

Proof. This problem is probably one of the best examples of Hölder's Inequality. It practically has Hölder's Inequality written all over it. First, we note that $3^3 = 27$, hence we might expect Hölder's Inequality to be used on the product of three terms. Next we note that,

$$2(a+b+c) = (a+b) + (b+c) + (c+a)$$

So, by multiplying both sides of the inequality by $2(a+b+c)^2$, it is equivalent with,

$$((a+b) + (b+c) + (c+a))(b+c+a)\left(\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)}\right) \geq 27$$

Which is true by Hölder's Inequality. Hence the inequality,

$$\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}$$

is also true, so we are done! □

Example 2.1.6: Let a and b be positive real numbers such that their sum is 1. Prove that

$$\frac{1}{a^2}+\frac{1}{b^2} \geq 8$$

Proof. First we note that,

$$\frac{1}{a^2}+\frac{1}{b^2}=(a+b)(a+b)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

Then, by Hölder's Inequality, we have,

$$(a+b)(a+b)\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \geq \left(\sqrt[3]{\frac{a\cdot a}{a^2}}+\sqrt[3]{\frac{b\cdot b}{b^2}}\right)^3=8$$

□

Example 2.1.7: Let a, b and c be positive real numbers such that $a+b+c=1$. Prove that

$$4a^3+9b^3+36c^3 \geq 1$$

Proof. Note that,

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$$

Then, by applying Hölder's Inequality, we have,

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)(4a^3+9b^3+36c^3) \geq (a+b+c)^3=1$$

And we're done. □

Example 2.1.8: Let a, b and c be positive real numbers. Prove that

$$\frac{a+b}{\sqrt{a+2c}} + \frac{b+c}{\sqrt{b+2a}} + \frac{c+a}{\sqrt{c+2b}} \geq 2\sqrt{a+b+c}$$

Proof. A common strategy used when solving problems that include square roots in the denominator is to square the expression on the left hand side then multiply by what's inside the square root times the numerator and apply Hölder's Inequality like so

$$\left(\sum_{cyc} \frac{a+b}{\sqrt{a+2c}}\right)^2 \left(\sum_{cyc} (a+b)(a+2c)\right) \geq 8(a+b+c)^3$$

Next we note that,

$$\sum_{cyc} (a+b)(a+2c) = (a+b+c)^2 + 3(ab+bc+ac)$$

Hence, it is sufficient to prove that

$$\frac{8(a+b+c)^3}{(a+b+c)^2 + 3(ab+bc+ac)} \geq (2\sqrt{a+b+c})^2$$

The rest of the proof is left as an exercise to the reader. □

2.1.1 Practice Problems

1. Let a, b and c be positive real numbers. Prove that

   (a)
   $$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{(a+b+c)^3}{3(ab+bc+ac)}$$

   (b)
   $$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \sqrt{\frac{27}{ab+bc+ac}}$$

(c)
$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\geq\frac{a+b+c}{2}$$

(d)
$$\frac{a^2+b^2+c^2}{a+b+c}\geq\sqrt{\frac{abc(a+b+c)}{ab+bc+ac}}$$

(e)
$$a^3+b^3+c^3\leq 3 \implies a+b+c\leq 3$$

2. Let a, b and c be positive real numbers such that $a+b+c=1$. Prove that

   (a)
   $$\sqrt[3]{99}\geq\sqrt[3]{1+8a}+\sqrt[3]{1+8b}+\sqrt[3]{1+8c}$$

   (b) For a positive integer n:
   $$\sqrt[n]{ab+bc+ac}\geq a\sqrt[n]{\frac{b+c}{2}}+b\sqrt[n]{\frac{a+c}{2}}+c\sqrt[n]{\frac{a+b}{2}}$$

3. Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Prove that
$$(1+a_1)(1+a_2)\cdots(1+a_n)\geq(1+\sqrt[n]{a_1a_2\cdots a_n})^n$$

4. Let a, b, c, x, y and z be positive real numbers. Prove that
$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z}\geq\frac{(a+b+c)^3}{3(x+y+z)}$$

5. Let a, b and c be positive real numbers such that $a+b+c=1$. Prove that
$$\frac{1}{a(3b+1)}+\frac{1}{b(3c+1)}+\frac{1}{c(3a+1)}\geq\frac{9}{2}$$

6. Let a and b be positive real numbers such that $a^2+b^2=1$. Prove that
$$\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{b}{a^2+1}+\frac{a}{b^2+1}\right)\geq\frac{8}{3}$$

7. Let a, b and c be positive real numbers. Prove that

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3} \leq \sqrt[3]{a \cdot \left(\frac{a+b}{2}\right) \cdot \left(\frac{a+b+c}{3}\right)}$$

8. (Vasile Cirtoaje) Let a, b and c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a+2b}}+\frac{b}{\sqrt{b+2c}}+\frac{c}{\sqrt{c+2a}} \geq \sqrt{a+b+c}$$

9. (Samin Riasat) Let a, b, c, m, n be positive real numbers. Prove that

$$\frac{a^2}{b(ma+nb)}+\frac{b^2}{c(mb+nc)}+\frac{c^2}{a(mc+na)} \geq \frac{3}{m+n}$$

10. (Indonesia, 2010) Let a, b and c be non-negative real numbers and let x, y and z be positive real numbers such that $a+b+c=x+y+z$. Prove that

$$\frac{a^3}{x^2}+\frac{b^3}{y^2}+\frac{c^3}{z^2} \geq a+b+c$$

11. (Greece, 2011) Let a, b, c be positive real numbers with sum 6. Find the maximum value of

$$S=\sqrt[3]{a^2+2bc}+\sqrt[3]{b^2+2ca}+\sqrt[3]{c^2+2ab}$$

12. (Junior Balkan Math Olympiad, 2011) Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$\prod_{cyc}(a^5+a^4+a^3+a^2+a+1) \geq 8(a^2+a+1)(b^2+b+1)(c^2+c+1)$$

13. (USAMO, 2004) For positive real numbers a, b and c. Prove that

$$(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) \geq (a+b+c)^3$$

14. (Austria, 2005) Let a, b, c and d be positive real numbers. Prove that

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3} \geq \frac{a+b+c+d}{abcd}$$

15. (Moldova TST, 2002) Positive numbers $\alpha, \beta, x_1, x_2, \cdots, x_n$ satisfy $x_1 + x_2 + \cdots + x_n = 1$ for all natural numbers n. Prove that

$$\frac{x_1^3}{\alpha x_1+\beta x_2}+\frac{x_2^3}{\alpha x_2+\beta x_3}+\cdots+\frac{x_n^3}{\alpha x_n+\beta x_1} \geq \frac{1}{n(\alpha+\beta)}$$

16. (IMO Longlist, 1986) Let k be one of the integers 2, 3, 4 and let $n = 2^k - 1$. Prove the inequality

$$1+b^k+b^{2k}+\cdots+b^{nk} \geq (1+b^n)^k$$

for all real $b \geq 0$.

17. (IMO Shortlist, 1998) Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+x)(1+z)}+\frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$

18. (IMO, 2001) Prove that for all positive real numbers a, b, c,

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \geq 1$$

2.1.2 Solutions

1. Let a, b and c be positive real numbers. Prove that

(a)
$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{(a+b+c)^3}{3(ab+bc+ac)}$$

Proof. By Hölder's Inequality, we have that

$$(1+1+1)(ab+bc+ac)\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)\geq(a+b+c)^3$$

and the result follows. □

(b)
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sqrt{\frac{27}{ab+bc+ac}}$$

Proof. By Hölder's Inequality, we have that

$$(ab+bc+ac)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)\geq 27$$

Moreover, this inequality is equivalent with

$$\begin{aligned}(ab+bc+ac)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 &\geq 27\\ \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 &\geq \frac{27}{ab+bc+ac}\\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c} &\geq \sqrt{\frac{27}{ab+bc+ac}}\end{aligned}$$

which is what we wanted to prove, so we are done! □

(c)
$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\geq\frac{a+b+c}{2}$$

Proof. By Hölder's Inequality, we have that

$$\left(\sum_{cyc} a+b\right)\left(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\right) \geq (a+b+c)^2$$

from where it follows that

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a} \geq \frac{a+b+c}{2}$$

□

(d)

$$\frac{a^2+b^2+c^2}{a+b+c} \geq \sqrt{\frac{abc(a+b+c)}{ab+bc+ac}}$$

Proof. Rearrange the inequality to its equivalent form

$$(ab+bc+ac)(a^2+b^2+c^2)^2 \geq abc(a+b+c)^3$$

divide both sides by abc

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a^2+b^2+c^2)^2 \geq (a+b+c)^3$$

and the result follows directly from Hölder's Inequality. □

(e)

$$a^3+b^3+c^3 \leq 3 \implies a+b+c \leq 3$$

Proof. Note that, by Hölder's Inequality, we have

$$27 \geq (1+1+1)(1+1+1)(a^3+b^3+c^3) \geq (a+b+c)^3$$

from where it's clear that

$$3 \geq a+b+c$$

which is what we wanted to prove, so we're done! □

2. Let a, b and c be positive real numbers such that $a+b+c=1$. Prove that

   (a)
   $$\sqrt[3]{99} \geq \sqrt[3]{1+8a}+\sqrt[3]{1+8b}+\sqrt[3]{1+8c}$$

   Proof. Note that
   $$99=(1+1+1)(1+1+1)((1+8a)+(1+8b)+(1+8c))$$
   and the result is evident. □

   (b) For a positive integer n:
   $$\sqrt[n]{ab+bc+ac} \geq a\sqrt[n]{\frac{b+c}{2}}+b\sqrt[n]{\frac{a+c}{2}}+c\sqrt[n]{\frac{a+b}{2}}$$

   Proof. Multiply both sides by $\sqrt[n]{2}$ so that our inequality is equivalent to
   $$\sqrt[n]{2ab+2bc+ac} \geq a\sqrt[n]{b+c}+b\sqrt[n]{a+c}+c\sqrt[n]{a+b}$$
   Then note that
   $$2ab+2bc+2ac=(a(b+c)+b(a+c)+c(a+b))(a+b+c)^{n-1}$$
   and the result is evident! □

3. Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Prove that
   $$(1+a_1)(1+a_2)\cdots(1+a_n) \geq (1+\sqrt[n]{a_1 a_2 \cdots a_n})^n$$

   Proof. It follows directly from Hölder's Inequality. □

4. Let a, b, c, x, y and z be positive real numbers. Prove that
   $$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)}$$

Proof. Multiply both sides by $3(x+y+z)$ and the result is evident! □

5. Let a, b and c be positive real numbers such that $a+b+c=1$. Prove that
$$\frac{1}{a(3b+1)}+\frac{1}{b(3c+1)}+\frac{1}{c(3a+1)} \geq \frac{9}{2}$$

Proof. By Hölder's Inequality, we have

$$\left(\sum_{cyc} a\right)\left(\sum_{cyc} 3b+1\right)\left(\frac{1}{a(3b+1)}+\frac{1}{b(3c+1)}+\frac{1}{c(3a+1)}\right) \geq 3^3$$

Thus, we have

$$\frac{1}{a(3b+1)}+\frac{1}{b(3c+1)}+\frac{1}{c(3a+1)} \geq \frac{27}{\left(\sum_{cyc} a\right)\left(\sum_{cyc} 3b+1\right)}$$

but since

$$(a+b+c)((3b+1)+(3c+1)+(3a+1))=(1)(6)$$

our inequality is equivalent to

$$\frac{1}{a(3b+1)}+\frac{1}{b(3c+1)}+\frac{1}{c(3a+1)} \geq \frac{9}{2}$$

which is what we wanted to prove, so we are done! □

6. Let a and b be positive real numbers such that $a^2+b^2=1$. Prove that

$$\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{b}{a^2+1}+\frac{a}{b^2+1}\right) \geq \frac{8}{3}$$

Proof. We start by multiplying both sides by 3

$$3\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{b}{a^2+1}+\frac{a}{b^2+1}\right) \geq 8$$

Then note that $3 = a^2 + b^2 + 1 + 1$ and that the inequality is equivalent to

$$([a^2+1]+[b^2+1])\left(\frac{1}{b}+\frac{1}{a}\right)\left(\frac{b}{a^2+1}+\frac{a}{b^2+1}\right) \geq (1+1)^3 = 8$$

which is what we wanted to prove, so we're done! □

7. Let a, b and c be positive real numbers. Prove that

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3} \leq \sqrt[3]{a\cdot\left(\frac{a+b}{2}\right)\cdot\left(\frac{a+b+c}{3}\right)}$$

Proof. This problem is all about being clever! Multiply both sides by 3 and the inequality is equivalent with

$$a+\sqrt{ab}+\sqrt[3]{abc} \leq \sqrt[3]{(a+a+a)\left(a+\frac{a+b}{2}+b\right)(a+b+c)}$$

Then note that, by Hölder's Inequality, we have

$$\sqrt[3]{(a+a+a)\left(a+\frac{a+b}{2}+b\right)\left(\sum_{cyc} a\right)} \geq a+\sqrt[3]{\frac{ab(a+b)}{2}}+\sqrt[3]{abc}$$

So it remains to prove that

$$\sqrt[3]{ab\left(\frac{a+b}{2}\right)} \geq \sqrt{ab}$$

which is equivalent to

$$\frac{a+b}{2} \geq \sqrt{ab}$$

and since this last inequality follows from the AM-GM inequality, we are done! □

8. (Vasile Cirtoaje) Let a, b and c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a+2b}}+\frac{b}{\sqrt{b+2c}}+\frac{c}{\sqrt{c+2a}} \geq \sqrt{a+b+c}$$

Proof. As we have done before, we square the left hand side and multiply by the corresponding terms. So that, by Hölder's Inequality, we have

$$\left(\sum_{cyc} \frac{a}{\sqrt{a+2b}}\right)^2 \left(\sum_{cyc} a(a+2b)\right) \geq (a+b+c)^3$$

Thus, it is sufficient to prove that

$$\frac{(a+b+c)^3}{\sum_{cyc} a(a+2b)} \geq a+b+c$$

which is clear since we have equality due to the fact that

$$\sum_{cyc} a(a+2b) = (a+b+c)^2$$

□

9. (Samin Riasat) Let a, b, c, m, n be positive real numbers. Prove that

$$\frac{a^2}{b(ma+nb)}+\frac{b^2}{c(mb+nc)}+\frac{c^2}{a(mc+na)} \geq \frac{3}{m+n}$$

Proof. Note that

$$\sum_{cyc} \frac{a^2}{b(ma+nb)} = \sum_{cyc} \frac{a^3}{ab(ma+nb)}$$

and, by Hölder's Inequality,

$$\left(\sum_{cyc} ab\right)\left(\sum_{cyc} ma+nb\right)\left(\sum_{cyc} \frac{a^3}{ab(ma+nb)}\right) \geq (a+b+c)^3$$

then, since

$$\sum_{cyc} ab = ab + bc + ac$$

and

$$\sum_{cyc} ma + nb = (m+n)(a+b+c)$$

the inequality is equivalent to

$$\sum_{cyc} \frac{a^3}{ab(ma+nb)} \geq \frac{(a+b+c)^3}{(m+n)(a+b+c)(ab+bc+ac)}$$

$$\frac{(a+b+c)^3}{(m+n)(a+b+c)(ab+bc+ac)} = \frac{(a+b+c)^2}{(m+n)(ab+bc+ac)}$$

$$\frac{(a+b+c)^2}{(m+n)(ab+bc+ac)} \geq \frac{3}{m+n}$$

In which we used the well-known inequality $(a+b+c)^2 \geq 3(ab+bc+ac)$ (which is equivalent to $(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$) thus our original inequality holds. □

10. (Indonesia, 2010) Let a, b and c be non-negative real numbers and let x, y and z be positive real numbers such that $a+b+c = x+y+z$. Prove that

$$\frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2} \geq a+b+c$$

Proof. By Hölder's Inequality it follows that

$$(x+y+z)(x+y+z)\left(\frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2}\right) \geq (a+b+c)^3$$

Thus, we have that

$$\frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2} \geq \frac{(a+b+c)^3}{(x+y+z)^2} = a+b+c$$

since $a+b+c = x+y+z$ and we're done! □

11. (Greece, 2011) Let a, b, c be positive real numbers with sum 6. Find the maximum value of

$$S = \sqrt[3]{a^2 + 2bc} + \sqrt[3]{b^2 + 2ca} + \sqrt[3]{c^2 + 2ab}$$

Proof. By Hölder's Inequality it follows that

$$(3)(3)\left(\sum_{cyc} a^2 + 2bc\right) \geq \left(\sum_{cyc} \sqrt[3]{a^2 + 2bc}\right)^3 = S^3$$

Furthermore,

$$(3)(3)((a^2 + 2bc) + (b^2 + 2ca) + (c^2 + 2ab)) = (3)(3)(a + b + c)^2 = 3^2 \cdot 6^2$$

so we have that

$$3^2 \cdot 6^2 \geq S^3$$

or

$$3\sqrt[3]{12} = \sqrt[3]{3^2 \cdot 6^2} \geq S$$

so the maximum is $\boxed{3\sqrt[3]{12}}$ and we have equality when a,b and c are equal which, with our condition, gives $a = b = c = 2$. □

12. (Junior Balkan Math Olympiad, 2011) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\prod_{cyc}(a^5 + a^4 + a^3 + a^2 + a + 1) \geq 8(a^2 + a + 1)(b^2 + b + 1)(c^2 + c + 1)$$

Proof. Let's start by noting that

$$a^5 + a^4 + a^3 + a^2 + a + 1 = (a^3 + 1)(a^2 + a + 1)$$

thus the problem is equivalent to proving that

$$(a^3 + 1)(b^3 + 1)(c^3 + 1) \geq 8$$

which follows from Hölder's Inequality as we have

$$(a^3 + 1)(b^3 + 1)(c^3 + 1) \geq (abc + 1)^3 = (1 + 1)^3 = 8$$

□

13. (USAMO, 2004) For positive real numbers a, b and c. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3$$

Proof. We notice that this inequality seems rather similar. In the first example we proved that

$$(a^3 + 2)(b^3 + 2)(c^3 + 2) \geq (a + b + c)^3$$

so it suffices to prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a^3 + 2)(b^3 + 2)(c^3 + 2)$$

Furthermore, if we can show that

$$x^5 - x^2 + 3 \geq x^3 + 2$$

then we're done. Luckily for us, this is true! Since

$$x^5 - x^2 + 3 \geq x^3 + 2 \iff (x^3 - 1)(x^2 - 1) \geq 0$$

and so we are done. □

14. (Austria, 2005) Let a, b, c and d be positive real numbers. Prove that

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} \geq \frac{a + b + c + d}{abcd}$$

Proof. Let $w = \frac{1}{a}$, $x = \frac{1}{b}$, $y = \frac{1}{c}$ and $z = \frac{1}{d}$ so that our inequality is equivalent with

$$w^3 + x^3 + y^3 + z^3 \geq xyz + wyz + wxz + wxy$$

Then note that, by Hölder's Inequality,

$$\left(\sum_{cyc} x^3\right)\left(\sum_{cyc} y^3\right)\left(\sum_{cyc} z^3\right) \geq \left(\sum_{cyc} xyz\right)^3$$

but we also have

$$(w^3 + x^3 + y^3 + z^3)^3 = \left(\sum_{cyc} x^3\right)\left(\sum_{cyc} y^3\right)\left(\sum_{cyc} z^3\right)$$

and the result follows immediately. □

15. (Moldova TST, 2002) Positive numbers $\alpha, \beta, x_1, x_2, \cdots, x_n$ satisfy $x_1 + x_2 + \cdots + x_n = 1$ for all natural numbers n. Prove that

$$\frac{x_1^3}{\alpha x_1 + \beta x_2} + \frac{x_2^3}{\alpha x_2 + \beta x_3} + \cdots + \frac{x_n^3}{\alpha x_n + \beta x_1} \geq \frac{1}{n(\alpha + \beta)}$$

Proof. We can rewrite the inequality like so

$$\sum_{cyc} \frac{x_1^3}{\alpha x_1 + \beta x_2} \geq \frac{1}{n(\alpha + \beta)}$$

Then, by Hölder's Inequality, we have

$$\left(\sum_{i=1}^{n} 1\right)\left(\sum_{cyc} (\alpha x_1 + \beta x_2)\right)\left(\sum_{cyc} \frac{x_1^3}{\alpha x_1 + \beta x_2}\right) \geq \left(\sum_{k=1}^{n} x_k\right)^3.$$

Then, noting that

$$\begin{aligned} \sum_{i=1}^{n} 1 &= n \\ \sum_{cyc} (\alpha x_1 + \beta x_2) &= (\alpha + \beta)(x_1 + x_2 + \cdots + x_n) = \alpha + \beta \\ \left(\sum_{k=1}^{n} x_k\right)^3 &= (1)^3 = 1 \end{aligned}$$

we have that the inequality is equivalent to

$$\begin{aligned} (n)(\alpha + \beta) \sum_{cyc} \frac{x_1^3}{\alpha x_1 + \beta x_2} &\geq 1 \\ \frac{x_1^3}{\alpha x_1 + \beta x_2} + \frac{x_2^3}{\alpha x_2 + \beta x_3} + \cdots + \frac{x_n^3}{\alpha x_n + \beta x_1} &\geq \frac{1}{n(\alpha + \beta)} \end{aligned}$$

which is what we wanted to prove. □

16. (IMO Longlist, 1986) Let k be one of the integers 2, 3, 4 and let $n = 2^k - 1$. Prove the inequality

$$1 + b^k + b^{2k} + \cdots + b^{nk} \geq (1 + b^n)^k$$

for all real $b \geq 0$.

Proof. Note that

$$1 + b^k + b^{2k} + \cdots + b^{nk} = (1 + b^k)(1 + b^{2k})(1 + b^{4k}) \cdots (1 + b^{2^{k-1}k})$$

Then, by Hölder's Inequality, we have

$$(1 + b^k)(1 + b^{2k})(1 + b^{4k}) \cdots (1 + b^{2^{k-1}k}) \geq (1 + b^{2^k - 1})^k = (1 + b^n)^k$$

and we're done! Note that this works for any $k \in \mathbb{N}$. $\square$

17. (IMO Shortlist, 1998) Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$

Proof. By Hölder's Inequality, we know that

$$\left(\sum_{cyc} (1+y)\right)\left(\sum_{cyc} (1+z)\right)\left(\sum_{cyc} \frac{x^3}{(1+y)(1+z)}\right) \geq (x+y+z)^3$$

Thus, it is sufficient to Prove that

$$\frac{(x+y+z)^3}{(3+x+y+z)^2} \geq \frac{3}{4}$$

or

$$(2(x+y+z))^2(x+y+z) \geq 3(3+x+y+z)^2$$

which follows from the fact that

$$x + y + z \geq 3\sqrt[3]{xyz} = 3$$

and

$$2(x+y+z) \geq 3 + (x+y+z)$$

and so we are done! $\square$

18. (IMO, 2001) Prove that for all positive real numbers a, b, c,

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1$$

Proof. By Hölder's Inequality, we know that

$$\left(\sum_{cyc}\frac{a}{\sqrt{a^2+8bc}}\right)^2\left(\sum_{cyc}a(a^2+8bc)\right)\geq (a+b+c)^3$$

so it suffices to show that

$$(a+b+c)^3\geq a^3+b^3+c^3+24abc$$

or its equivalent form

$$a(b-c)^2+b(c-a)^2+c(a-b)^2\geq 0$$

which obviously holds true, thus our original inequality also holds true. □

2.2 Induction

When we work with induction we always check if the base case (first case) works, assume that the statement is true for some n and then prove for $n+1$. The reasoning for why this works to prove all cases should be intuitive (consider each case as if it were part of a long line of dominoes where the previous domino hits the one immediately after).

We'll begin this section with an example from Romania's National Math Olympiad in 2008. We should note that this inequality hints us to use induction as the terms of the previous cases still remain in the later cases. This allows us to apply the inequalities of the lower cases to prove the larger ones.

Example 2.2.1: (Romania, 2008) Prove that

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{2n}}>n$$

for all positive integers n.

Proof. Let's first start by checking if $n=1$ works

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}>\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1$$

and so it does. Now let us assume that the statement is true for some n

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{2n}}>n$$

This tells us that

$$\frac{1}{2}+\cdots+\frac{1}{2^{2n}}+\frac{1}{2^{2n}+1}+\cdots+\frac{1}{2^{2(n+1)}}>n+\frac{1}{2^{2n}+1}+\cdots+\frac{1}{2^{2(n+1)}}$$

Furthermore, we know that

$$\sum_{k=1}^{2^{2n}}\frac{1}{2^{2n}+k}=\frac{1}{2^{2n}+1}+\frac{1}{2^{2n}+2}+\cdots+\frac{1}{2^{2n+1}}>\frac{2^{2n}}{2^{2n+1}}=\frac{1}{2}$$

since

$$\frac{1}{2^{2n}+k} > \frac{1}{2^{2n+1}} \text{ for } 1 \leq k \leq 2^{2n} - 1$$

similarly

$$\sum_{k=1}^{2^{2n+1}} \frac{1}{2^{2n+1}+k} = \frac{1}{2^{2n+1}+1} + \frac{1}{2^{2n+1}+2} + \cdots + \frac{1}{2^{2n+2}} > \frac{2^{2n+1}}{2^{2n+2}} = \frac{1}{2}$$

joining these two inequalities we get

$$\left(\sum_{k=1}^{2^{2n}} \frac{1}{2^{2n}+k}\right) + \left(\sum_{k=1}^{2^{2n+1}} \frac{1}{2^{2n+1}+k}\right) > \frac{1}{2} + \frac{1}{2} = 1$$

returning to

$$\frac{1}{2} + \cdots + \frac{1}{2^{2n}} + \frac{1}{2^{2n}+1} + \cdots + \frac{1}{2^{2(n+1)}} > n + \frac{1}{2^{2n}+1} + \cdots + \frac{1}{2^{2(n+1)}}$$

we notice that

$$\frac{1}{2^{2n}+1} + \cdots + \frac{1}{2^{2(n+1)}} = \left(\sum_{k=1}^{2^{2n}} \frac{1}{2^{2n}+k}\right) + \left(\sum_{k=1}^{2^{2n+1}} \frac{1}{2^{2n+1}+k}\right)$$

and thus

$$\frac{1}{2} + \cdots + \frac{1}{2^{2n}} + \frac{1}{2^{2n}+1} + \cdots + \frac{1}{2^{2(n+1)}} > n + \frac{1}{2^{2n}+1} + \cdots + \frac{1}{2^{2(n+1)}} > n+1.$$

This in turn proves the $n+1$ case for our inequality and so we're done. □

The next example is from the Ibero American Math Olympiad that was held in Costa Rica in September, 2011.

Example 2.2.2: (Ibero, 2011) Let $x_1, \cdots, x_n$ be positive real numbers. Show that there exist $a_1, \cdots, a_n \in \{-1, 1\}$ such that:

$$a_1 x_1^2 + a_2 x^2 + \cdots + a_n x_n^2 \geq (a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)^2$$

Proof. Let's start by noting that we can assume that $x_1 \geq x_2 \geq \cdots \geq x_n$ since the coefficients (i.e. the a_i's) are arbitrary. Next, we note that the following inequalities hold

$$a^2 - b^2 \geq (a-b)^2 \text{ when } a \geq b$$

and

$$a^2 - b^2 + c^2 \geq (a-b+c)^2 \text{ when } a \geq b \geq c$$

since it's equivalent with

$$(a-b)(b-c) \geq 0.$$

From here we can conjecture that $a_k = (-1)^{k+1}$. Now we'll prove (by Induction) the inequality for $n = 2k-1$, $k \in \mathbb{N}$ and $a_k = (-1)^{k+1}$. For $k = 1 \implies n = 1$ it clearly holds. Let's assume it's true for some k.

$$x_1^2 - x_2^2 + \cdots + x_{2k-1}^2 \geq (x_1 - x_2 + \cdots + x_{2k-1})^2$$

and so we need to prove

$$x_1^2 - x_2^2 + \cdots + x_{2k-1}^2 - x_{2k}^2 + x_{2k+1}^2 \geq (x_1 - x_2 + \cdots + x_{2k+1})^2$$

using our hypothesis we only need to prove that

$$(x_1 - x_2 + \cdots + x_{2k-1})^2 - x_{2k}^2 + x_{2k+1}^2 \geq (x_1 - x_2 + \cdots + x_{2k+1})^2$$

If we let $a = x_1 - x_2 + \cdots + x_{2k-1}$, $b = x_{2k}$ and $c = x_{2k+1}$ then we can easily note that $a \geq b \geq c$, thus

$$a^2 - b^2 + c^2 \geq (a-b+c)^2$$

which is equivalent with

$$(x_1 - x_2 + \cdots + x_{2k-1})^2 - x_{2k}^2 + x_{2k+1}^2 \geq (x_1 - x_2 + \cdots + x_{2k+1})^2$$

and so we have proven the n odd case. To prove the n even case (i.e. $n = 2k$ for all $k \in \mathbb{N}$.) we use the fact that n odd is already proven so

$$x_1^2 - x_2^2 + \cdots + x_{2k-1}^2 - x_{2k}^2 \geq (x_1 - x_2 + \cdots + x_{2k-1})^2 - x_{2k}^2$$

since $x_1 - x_2 + \cdots + x_{2k-1} \geq x_{2k}$ we have

$$(x_1 - x_2 + \cdots + x_{2k-1})^2 - x_{2k}^2 \geq (x_1 - x_2 + \cdots + x_{2k-1} - x_{2k})^2$$

or

$$x_1^2 - x_2^2 + \cdots + x_{2k-1}^2 - x_{2k}^2 \geq (x_1 - x_2 + \cdots + x_{2k-1} - x_{2k})^2.$$

Thus, our conjecture that $a_k = (-1)^{k+1}$ would give us such numbers is true and we're done. □

Sometimes this method of mathematical induction does not suffice and we require a stronger argument. This stronger form of mathematical induction requires that the we assume that the first n cases are true rather than just the n^{th} case. Note that this argument still holds intuitively.

Example 2.2.3: (APMO, 1999) The real numbers $a_1, a_2, a_3, ...$ satisfy $a_{i+j} \leq a_i + a_j$ for all i, j. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n$$

Proof. (By strong induction)

The base case, $n = 1$, follows immediately

$$a_1 \geq a_1.$$

Next, we may assume that for some n the inequality holds for all $k \in \mathbb{N}$ such that $1 \leq k \leq n$:

$$\begin{aligned} a_1 &\geq a_1 \\ a_1 + \frac{a_2}{2} &\geq a_2 \\ &\vdots \\ a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n} &\geq a_n \end{aligned}$$

Adding these inequalities together we get

$$na_1 + \frac{(n-1)a_2}{2} + \frac{(n-2)a_3}{3} + \cdots + \frac{(1)a_n}{n} \geq a_1 + a_2 + \cdots + a_n.$$

Then, by adding $a_1 + a_2 + \cdots + a_n$ to both sides, we have

$$(n+1)a_1 + \frac{(n+1)a_2}{2} + \cdots + \frac{(n+1)a_n}{n} \geq (a_1 + a_n) + (a_2 + a_{n-1}) + \cdots + (a_n + a_1).$$

From where it follows that

$$(n+1)\left(a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n}\right) \geq na_{n+1}$$

or its equivalent

$$a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n} + \frac{a_{n+1}}{n+1} \geq a_{n+1}.$$

Which concludes the induction. □

2.2.1 Practice Problems

Try to solve the small cases of these problems to get a sense of how to do the induction.

1. Let $n \geq 2$ be a natural number. Prove the following inequality

$$3^n > 3n + 2$$

2. For $n \geq 4$ a natural number, Prove that

$$2^n < n!$$

3. Let a_1 be a positive real number such that $a_1 < \frac{1}{2}$. Given that $a_{n+1} = 2a_n^3 + a_n^2$ for all $n \geq 1$. Prove that

$$a_n < \frac{1}{2} \; \forall n \in \mathbb{N}$$

4. (Complex Triangle Inequality Generalization) Let $x_1, x_2, \cdots, x_n$ be complex numbers. Prove that

$$|x_1| + |x_2| + \cdots + |x_n| \geq |x_1 + x_2 + \cdots + x_n|$$

5. (Bernoulli's Inequality) Prove that for all natural numbers $n > 1$ and real numbers $x > -1$ we have

$$(1+x)^n \geq 1 + nx$$

6. Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be two sequences of real numbers. Prove that

$$\sqrt{a_1^2 + b_1^2} + \cdots + \sqrt{a_n^2 + b_n^2} \geq \sqrt{(a_1 + \cdots + a_n)^2 + (b_1 + \cdots + b_n)^2}$$

7. Let $a_n \geq a_{n-1} \geq \cdots \geq a_1$ be positive real numbers and $n \geq 2$ a natural number. Prove that

$$a_n^2 - a_1^2 \geq (a_n - a_{n-1})^2 + (a_{n-1} - a_{n-2})^2 + \cdots + (a_2 - a_1)^2$$

8. Let n be a natural number. Prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$$

9. (Romania District Olympiad, 2001) Consider a positive odd integer k and let $n_1 < n_2 < \cdots < n_k$ be k positive odd integers. Prove that

$$n_1^2 - n_2^2 + n_3^2 - n_4^2 + \cdots + n_k^2 \geq 2k^2 - 1$$

2.2.2 Solutions

1. Let $n \geq 2$ be a natural number. Prove the following inequality

$$3^n > 3n + 2$$

Proof. For $n = 2$ the inequality clearly holds. We assume that the inequality holds for n and prove for $n + 1$.

$$3^{n+1} = 3 \cdot 3^n > 3(3n + 2) > 3(n + 1) + 2$$

Thus, we have that

$$3^n > 3n + 2 \implies 3^{n+1} > 3(n + 1) + 2$$

and so we are done. □

2. For $n \geq 4$ a natural number, Prove that

$$2^n < n!$$

Proof. For $n = 4$ we see that it holds true. Next we assume it holds for n and note that $n + 1 > 2$ for all $n \geq 4$. So we have

$$(n + 1)! = (n + 1) \cdot n! > (n + 1) \cdot 2^n > (2) \cdot 2^n > 2^{n+1}$$

and so

$$n! > 2^n \implies (n + 1)! > 2^{n+1}$$

which is what we wanted to prove so we're done. □

3. Let a_1 be a positive real number such that $a_1 < \frac{1}{2}$. Given that $a_{n+1} = 2a_n^3 + a_n^2$ for all $n \geq 1$. Prove that

$$a_n < \frac{1}{2} \; \forall n \in \mathbb{N}$$

Proof. For $n = 2$ we have

$$a_2 = 2a_1^3 + a_1^2 < 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

thus it holds for $n = 2$. Furthermore, if we assume that $a_n < \frac{1}{2}$ for some n then

$$a_{n+1} = 2a_n^3 + a_n^2 < 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

which concludes the induction process. □

4. (Complex Triangle Inequality Generalization) Let $x_1, x_2, \cdots, x_n$ be complex numbers. Prove that

$$|x_1| + |x_2| + \cdots + |x_n| \geq |x_1 + x_2 + \cdots + x_n|$$

Proof. For $n = 2$ we note that [in the complex plane] $|x_1 + x_2|$ is the length of a diagonal in a parallelogram with sides $|x_1|$ and $|x_2|$ thus by the triangle inequality we have

$$|x_1| + |x_2| \geq |x_1 + x_2|$$

Furthermore, if we assume that

$$|x_1| + |x_2| + \cdots + |x_n| \geq |x_1 + x_2 + \cdots + x_n|$$

is true then

$$|x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}| \geq |x_1 + x_2 + \cdots + x_n| + |x_{n+1}|$$

but from the $n = 2$ case we have

$$|x_1 + x_2 + \cdots + x_n| + |x_{n+1}| \geq |x_1 + x_2 + \cdots + x_n + x_{n+1}|$$

which proves the $n + 1$ case so we're done. □

5. (Bernoulli's Inequality) Prove that for all natural numbers $n > 1$ and real numbers $x > -1$ we have

$$(1+x)^n \geq 1 + nx$$

Proof. For $n = 2$ we have

$$(1+x)^2 = 1 + 2x + x^2 \geq 1 + 2x$$

because $x^2 \geq 0$. Next we assume that it's true for some n

$$(1+x)^n \geq 1 + nx$$

then for $n+1$ we have

$$(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx)$$

and by expanding

$$(1+x)(1+nx) = 1 + (n+1)x + nx^2 \geq 1 + (n+1)x$$

which proves the $n+1$ case and so the problem solved. □

6. Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be two sequences of real numbers. Prove that

$$\sqrt{a_1^2 + b_1^2} + \cdots + \sqrt{a_n^2 + b_n^2} \geq \sqrt{(a_1 + \cdots + a_n)^2 + (b_1 + \cdots + b_n)^2}$$

Proof. Let the complex numbers $x_k = a_k + i \cdot b_k$ for all $1 \leq k \leq n$. Then note that the inequality is equivalent to

$$|x_1| + |x_2| + \cdots + |x_n| \geq |x_1 + x_2 + \cdots + x_n|$$

which was solved as problem 4. □

7. Let $a_n \geq a_{n-1} \geq \cdots \geq a_1$ be positive real numbers and $n \geq 2$ a natural number. Prove that

$$a_n^2 - a_1^2 \geq (a_n - a_{n-1})^2 + (a_{n-1} - a_{n-2})^2 + \cdots + (a_2 - a_1)^2$$

Proof. For $n = 2$ we have

$$a_2^2 - a_1^2 \geq (a_2 - a_1)^2 \iff a_2 \geq a_1$$

and so it holds. We assume the inequality holds for some n such that $a_n \geq a_{n-1} \geq \cdots \geq a_1$

$$a_n^2 - a_1^2 \geq (a_n - a_{n-1})^2 + (a_{n-1} - a_{n-2})^2 + \cdots + (a_2 - a_1)^2$$

then for some a_{n+1} such that $a_{n+1} \geq a_n$ we have

$$a_{n+1}^2 - a_n^2 \geq (a_{n+1} - a_n)^2$$

by the $n = 2$ case. Adding this inequality to the n case we have

$$(a_{n+1}^2 - a_n^2) + (a_n^2 - a_1^2) \geq (a_{n+1} - a_n)^2 + (a_n - a_{n-1})^2 + \cdots + (a_2 - a_1)^2$$

which is equivalent to

$$a_{n+1}^2 - a_1^2 \geq (a_{n+1} - a_n)^2 + (a_n - a_{n-1})^2 + \cdots + (a_2 - a_1)^2$$

thus, since this is the $n+1$ case, we have that the inequality then follows for all $n \geq 2$. □

8. Let n be a natural number. Prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$$

Proof. We will prove a stronger inequality:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

For $n = 1$ we have

$$1 \leq 2 - 1 = 1$$

and so the base case holds. Then, for some n we assume that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

adding $\frac{1}{(n+1)^2}$ to both sides we get

$$1+\frac{1}{2^2}+\cdots+\frac{1}{(n+1)^2}\leq 2-\frac{1}{n}+\frac{1}{(n+1)^2}$$

and so it is sufficient to prove that

$$2-\frac{1}{n}+\frac{1}{(n+1)^2}\leq 2-\frac{1}{n+1}$$

or

$$\begin{aligned}\frac{1}{(n+1)^2} &\leq \frac{1}{n}-\frac{1}{n+1}\\ \frac{1}{(n+1)^2} &\leq \frac{1}{n(n+1)}\\ n &\leq n+1\end{aligned}$$

which of course holds for all natural numbers n and so the problem is solved since $2-\frac{1}{n}<2$. Alternatively, we could have noted that

$$1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}<\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}<2$$

The identity used, $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$, is from the Basel Problem. □

9. (Romania District Olympiad, 2001) Consider a positive odd integer k and let $n_1<n_2<\cdots<n_k$ be k positive odd integers. Prove that

$$n_1^2-n_2^2+n_3^2-n_4^2+\cdots+n_k^2\geq 2k^2-1$$

Proof. First we should note that $n_q\geq 2q-1$ for all $1\leq q\leq k$ since for all $1\leq p\leq q-1$ we have that n_p are odd numbers smaller than n_q and so n_q should be larger than the $(q-1)$th odd number. For $k=1$ we have

$$n_1^2\geq 2\cdot 1^2-1=1$$

and so the base case holds. We assume the inequality holds for some k

$$n_1^2 - n_2^2 + n_3^2 - n_4^2 + \cdots + n_k^2 \geq 2k^2 - 1$$

and note that the next case is $k+2$ (as it needs to remain odd). If n_{k+2} and n_{k+1} are odd integers such that $n_{k+2} > n_{k+1} > n_k > \cdots > n_1$ then we need to prove that

$$n_1^2 - n_2^2 + n_3^2 - n_4^2 + \cdots + n_{k+2}^2 \geq 2(k+2)^2 - 1.$$

Using the k case, we have that it is sufficient to prove that

$$n_{k+2}^2 - n_{k+1}^2 \geq (2(k+2)^2 - 1) - (2k^2 - 1)$$

or

$$\begin{aligned} n_{k+2}^2 - n_{k+1}^2 &\geq 8k+8 \\ (n_{k+2} + n_{k+1})(n_{k+2} - n_{k+1}) &\geq 8k+8 \end{aligned}$$

Note that the left hand side expression is minimized when both $n_{k+2} + n_{k+1}$ and $n_{k+2} - n_{k+1}$ are minimized. The smallest value for $n_{k+2} - n_{k+1}$ is 2 and it happens when n_{k+2} and n_{k+1} are consecutive (odd integers). Furthermore, the smallest value for $n_{k+2} + n_{k+1}$ happens when n_{k+2} and n_{k+1} are minimized which happens when $n_{k+2} = 2k+3$ and $n_{k+1} = 2k+1$. Since these values are consecutive we have that the expression is minimized and so

$$n_{k+2}^2 - n_{k+1}^2 \geq (2k+3)^2 - (2k+1)^2 = 8k+8$$

which is what we wanted to prove, so we are done. □

2.3 Schur's Inequality

Schur's Inequality is special for its equality cases. When an inequality has unusual cases for equality you might want to try using Schur's Inequality.

Theorem 2.3.1 (Schur's Inequality)**:** Let a, b, c be nonnegative reals and $r > 0$. Then

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0$$

with equality if and only if $a = b = c$ or some two of a, b, c are equal and the other is 0.

Proof. We can assume without loss of generality that $a \geq b \geq c$ since the inequality is symmetric. Now note that

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) = (a-b)\left(a^r(a-c) - b^r(b-c)\right)$$

then since $a - b \geq 0$, $a^r \geq b^r \geq 0$ and $a - c \geq b - c \geq 0$ it's clear that

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) \geq 0$$

Furthermore, we have that

$$c^r(c-a)(c-b) \geq 0$$

since $c^r \geq 0$ and $(c-a)(c-b) = (a-c)(b-c) \geq 0$. By adding these two inequalities together we get

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0$$

□

We'll start by proving an equivalent form of the $r = 1$ case.

Example 2.3.2: Let a, b and c be nonnegative real numbers. Prove that

$$a^3 + b^3 + c^3 + 3abc \geq a^2(b+c) + b^2(a+c) + c^2(a+b)$$

Proof. Rewrite the inequality as

$$a(a^2 + bc) + b(b^2 + ac) + c(c^2 + ab) \geq a(ab + ac) + b(ba + bc) + c(ca + cb)$$

then move the terms that are on the right to the left

$$a(a^2 - ab - ac + bc) + b(b^2 - ab - bc + ac) + c(c^2 - ac - bc + ab) \geq 0$$

which is equivalent to

$$a(a - b)(a - c) + b(b - c)(b - a) + c(c - a)(c - b) \geq 0$$

Since this is the $r = 1$ case of Shur's Inequality we have the inequality is proven. □

Note that if you can simplify an inequality problem to any equivalent form of Schur's Inequality then you have just proved that inequality (assuming you know how the inequality is equivalent).

Example 2.3.3: Let a, b and c be nonnegative real numbers. Prove that

$$abc \geq (a + b - c)(a + c - b)(b + c - a)$$

Proof. By expanding the expression, we have

$$abc \geq a^2(b + c) + b^2(a + c) + c^2(a + b) - 2abc - (a^3 + b^3 + c^3)$$

or

$$a^3 + b^3 + c^3 + 3abc \geq a^2(b + c) + b^2(a + c) + c^2(a + b)$$

Which we know is equivalent to Schur's Inequality for $r = 1$ so we're done. □

Sometimes you'll have to make several algebraic manipulations to check if an inequality can be solved by Schur's Inequality. This is the case with the next example.

Example 2.3.4: (British Mathematical Olympiad, 1999) Nonnegative real numbers p, q and r satisfy $p + q + r = 1$. Prove that

$$7(pq + qr + rp) \leq 2 + 9pqr$$

Proof. We note that the left hand side has degree 2 while the right hand side has a term of degree 0 and another of degree 3. We can homogenize this inequality by using the condition given like so

$$7(pq+qr+rp)(p+q+r) \leq 2(p+q+r)^3 + 9pqr$$

Now all the terms have the same degree. Then we note the following identity

$$(p+q+r)^3 + 3pqr = p^3+q^3+r^3+3(p+q+r)(pq+qr+rp)$$

Thus, the given inequality is equivalent to

$$2(p^3+q^3+r^3)+3pqr \geq (p+q+r)(pq+qr+rp)$$

we subtract $3pqr$ from both sides and get

$$2(p^3+q^3+r^3) \geq p^2(q+r)+q^2(p+r)+r^2(p+q)$$

which looks very similar to Schur's Inequality by excercise 2.3.2. We know that

$$p^3+q^3+r^3+3pqr \geq p^2(q+r)+q^2(p+r)+r^2(p+q)$$

holds true. So it is sufficient to prove that

$$p^3+q^3+r^3 \geq 3pqr$$

But this follows from the AM-GM inequality, so we're done. □

2.3.1 Practice Problems

1. Let a, b and c be nonnegative real numbers. Prove that

$$(a^2+b^2+c^2)(a+b+c)+7abc \geq 2(a+b)(a+c)(b+c)$$

2. Let a, b and c be nonnegative real numbers such that $a+b+c=1$. Prove that

$$2(a^3+b^3+c^3)+3abc \geq a^2+b^2+c^2$$

3. (IMO, 1964) Denote by a, b, c the lengths of the sides of a triangle. Prove that
$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) \leq 3abc$$

4. Let a, b, c be positive real numbers such that $abc = 1$. Prove that
$$a^3+b^3+c^3 \geq \frac{(a-b)(b-c)(c-a)}{2}+\frac{a}{c}+\frac{b}{a}+\frac{c}{b}$$

5. (Canada MO, 1992) For $x, y, z \geq 0$, establish the inequality
$$x(x-z)^2+y(y-z)^2 \geq (x-z)(y-z)(x+y-z)$$
and determine when equality holds.

6. Let a, b and c be nonnegative real numbers such that $a+b+c=1$. Prove that
$$9abc+1 \geq 4(ab+bc+ac)$$

7. (Austrian-Polish Mathematical Competition, 2001) If a, b, c are the sides of a triangle, prove the inequality
$$2 < \frac{a+b}{c}+\frac{a+c}{b}+\frac{b+c}{a}-\frac{a^3+b^3+c^3}{abc} \leq 3$$

8. (IMO, 1984) Let x, y, z be nonnegative real numbers with $x+y+z=1$. Show that
$$0 \leq xy+yz+zx-2xyz \leq \frac{7}{27}$$

2.3.2 Solutions

1. Let a, b and c be nonnegative real numbers. Prove that

$$(a^2 + b^2 + c^2)(a + b + c) + 7abc \geq 2(a + b)(a + c)(b + c)$$

Proof. By expanding and simplifying we note that the inequality is equivalent with

$$a^3 + b^3 + c^3 + 3abc \geq a^2(b + c) + b^2(a + c) + c^2(a + b)$$

which we know holds since it is equivalent to Schur's Inequality for $r = 1$. □

2. Let a, b and c be nonnegative real numbers such that $a+b+c = 1$. Prove that

$$2(a^3 + b^3 + c^3) + 3abc \geq a^2 + b^2 + c^2$$

Proof. Subtract $a^3 + b^3 + c^3$ on both sides to get

$$\begin{aligned} a^3 + b^3 + c^3 + 3abc &\geq a^2 + b^2 + c^2 - (a^3 + b^3 + c^3) \\ a^3 + b^3 + c^3 + 3abc &\geq a^2(1 - a) + b^2(1 - b) + c^2(1 - c) \\ a^3 + b^3 + c^3 + 3abc &\geq a^2(b + c) + b^2(a + c) + c^2(a + b) \end{aligned}$$

which follows from Schur's Inequality. □

3. (IMO, 1964) Denote by a, b, c the lengths of the sides of a triangle. Prove that

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc$$

Proof. After expanding and rearranging we note that this inequality is equivalent to

$$a^3 + b^3 + c^3 + 3abc \geq a^2(b + c) + b^2(a + c) + c^2(a + b)$$

which follows from Schur's Inequality for $r = 1$. Alternatively, we could note that the inequality is equivalent to

$$abc \geq (a+b-c)(a+c-b)(b+c-a)$$

which, after using Ravi Substitution[1], is equivalent to

$$(y+z)(x+z)(x+y) \geq 8xyz$$

but this last inequality follows from AM-GM so we're done. □

4. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$a^3+b^3+c^3 \geq \frac{(a-b)(b-c)(c-a)}{2} + \frac{a}{c} + \frac{b}{a} + \frac{c}{b}$$

Proof. To begin, we use the condition to note that inequality is equivalent to

$$a^3+b^3+c^3 \geq \frac{(a-b)(b-c)(c-a)}{2} + a^2b + b^2c + c^2a$$

Next, we multiply by 2 on both sides. By expanding the expression $(a-b)(b-c)(c-a)$ and simplifying, we have that the inequality is equivalent to

$$2(a^3+b^3+c^3) \geq a^2(b+c) + b^2(a+c) + (a+b)^2.$$

From Schur's Inequality we have that

$$a^3+b^3+c^3+3abc \geq a^2(b+c) + b^2(a+c) + (a+b)^2.$$

So it suffices to show the following inequality

$$\begin{aligned} 2(a^3+b^3+c^3) &\geq a^3+b^3+c^3+3abc \\ a^3+b^3+c^3 &\geq 3abc. \end{aligned}$$

Since this inequality follows from the AM-GM inequality, we're done! □

[1] Ravi Substitution: Is the process in which we express the side lengths of a triangle in terms of the distances from its vertices to the two nearest tangency points of the incircle.

5. (Canada MO, 1992) For $x, y, z \geq 0$, establish the inequality

$$x(x-z)^2 + y(y-z)^2 \geq (x-z)(y-z)(x+y-z)$$

and determine when equality holds.

Proof. By expanding and simplifying we can rearrange the inequality so that it represents Schur's Inequality for $r = 1$. Therefore, the inequality is proven and has equality cases when $x = y = z$ or two of x, y and z are equal and the last one is 0. □

6. Let a, b and c be nonnegative real numbers such that $a+b+c = 1$. Prove that

$$9abc + 1 \geq 4(ab + bc + ac)$$

Proof. First we note that this inequality has equality when $a = b = c = \frac{1}{3}$ or one of a, b, c is 0 and the other two are $\frac{1}{2}$. This hints to us that we should look for Schur's Inequality. Then we, as done in a similar example problem, plug in the given condition so that all terms have the same degree.

$$9abc + (a+b+c)^3 \geq 4(ab + bc + ac)(a+b+c)$$

And note that, after expansion and simplification, this is equivalent to Schur's Inequality for $r = 1$. □

7. (Austrian-Polish Mathematical Competition, 2001) If a, b, c are the sides of a triangle, prove the inequality

$$2 < \frac{a+b}{c} + \frac{a+c}{b} + \frac{b+c}{a} - \frac{a^3+b^3+c^3}{abc} \leq 3$$

Proof. First we'll prove the left hand side

$$\begin{aligned} 2 &< \frac{a+b}{c} + \frac{a+c}{b} + \frac{b+c}{a} - \frac{a^3+b^3+c^3}{abc} \\ 2abc &< ab(a+b) + ac(a+c) + bc(b+c) - (a^3+b^3+c^3) \end{aligned}$$

By subtracting $2abc$ to both sides and factorizing we get

$$\begin{aligned} 0 &< ab(a+b)+ac(a+c)+bc(b+c)-(a^3+b^3+c^3)-2abc \\ 0 &< (a+b-c)(a+c-b)(b+c-a) \end{aligned}$$

which follows from the triangle inequality. To prove the right hand side we multiply both sides by abc

$$ab(a+b)+ac(a+c)+bc(b+c)-(a^3+b^3+c^3) \leq 3abc$$

or

$$ab(a+b)+ac(a+c)+bc(b+c) \leq a^3+b^3+c^3+3abc$$

but

$$ab(a+b)+ac(a+c)+bc(b+c) = a^2(b+c)+b^2(a+c)+c^2(a+b)$$

now we note that our inequality is equivalent to Schur's Inequality for $r=1$ so we're done. □

8. (IMO, 1984) Let x, y, z be nonnegative real numbers with $x+y+z=1$. Show that

$$0 \leq xy+yz+zx-2xyz \leq \frac{7}{27}$$

Proof. We'll start with the left hand side. Notice that for all real x, y, z we have that

$$(x+y+z)(xy+yz+zx)-2xyz = xyz$$

So the left hand side is equivalent to proving that

$$xyz \geq 0$$

which follows from the conditions. For the right hand side we use the condition to make all of the terms have the same degree.

$$\begin{aligned}
xy + yz + zx - 2xyz &\leq \frac{7}{27} \\
(x + y + z)(xy + yz + zx) - 2xyz &\leq \frac{7(x + y + z)^3}{27} \\
x^2(y + z) + y^2(x + z) + z^2(x + y) + xyz &\leq \frac{7(x + y + z)^3}{27} \\
27\left(x^2(y + z) + y^2(x + z) + z^2(x + y) + xyz\right) &\leq 7(x + y + z)^3
\end{aligned}$$

which simplifies to

$$6\left(x^2(y + z) + y^2(x + z) + z^2(x + y)\right) \leq 7(x^3 + y^3 + z^3) + 15xyz$$

Finally, we note that

$$7(x^3 + y^3 + z^3) + 15xyz \geq 6(x^3 + y^3 + z^3 + 3xyz)$$

since it's equivalent to

$$x^3 + y^3 + z^3 \geq 3xyz$$

which follows from the AM-GM inequality. So our original inequality is equivalent with proving

$$x^3 + y^3 + z^3 + 3xyz \geq x^2(y + z) + y^2(x + z) + z^2(x + y)$$

which is Schur's Inequality for $r = 1$ so we're done. □

Notation

$\mathbb{N}$	The set of positive integers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of positive real numbers
$\forall$	For all
$\in$	In
$\lvert x \rvert$	The absolute value of x
$\sum_{cyc} f(a_1, \cdots)$	The sum of the function f applied cyclically
$\prod_{cyc} f(a_1, \cdots)$	The product of the function f applied cyclically
$\max\{a_1, a_2, \cdots\}$	The largest element in the set $\{a_1, a_2, \cdots\}$
$\min\{a_1, a_2, \cdots\}$	The smallest element in the set $\{a_1, a_2, \cdots\}$
IMO	International Math Olympiad
ISL	International Math Olympiad Short-List
IBERO	Ibero American Math Olympiad
CENTRO	Central American and Caribbean Math Olympiad
APMO	Asian Pacific Mathematics Olypmiad
USAMO	United States of America Mathematical Olympiad
TST	Team Selection Test

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